ISRAEL JOURNAL OF MATHEMATICS **96** (1996), 141–183

DECOMPOSABILITY AND EMBEDDABILITY OF DISCRETELY HENSELIAN DIVISION ALGEBRAS

BY

ERIC S. BRUSSEL*

Department of Mathematics, Harvard University
Cambridge, MA 02138, USA
e-mail: brussel@abel.math.harvard.edu

Dedicated to the memory of Amitsur

ABSTRACT

Let F be a discretely Henselian field of rank one, with residue field k a number field, and let D/F be an F-division algebra. We conduct an exhaustive study of the decomposability of an arbitrary D. Specifically, we prove the following: D has a semiramified (SR) F-division subalgebra if and only if D has a totally ramified (TR) subfield. However, there may be TR subfields not contained in any SR subalgebra. If D has prime-power index, then D is decomposable if and only if D properly contains a SR division subalgebra. Equivalently, D has a decomposable Sylow factor if and only if $\mathfrak{i}(D^{\otimes n}) \neq \frac{1}{n}\,\mathfrak{i}(D)$ for some n dividing the period of D, that is, if and only if the index fails to mimic the behavior of the period of D. There exists indecomposable D with prime-power period p^2 and index p^3. Every proper division subalgebra of D is indecomposable. Conversely, every indecomposable F-division algebra of p-power index embeds properly in some D of p-power index if and only if k does not have a certain strengthened form of class field theory's Special Case. Semiramified division algebras and division algebras of odd index always properly embed. Finally, these results apply to an extent over $k(t)$, and we prove that there exist indecomposable $k(t)$-division algebras of period p^2 and index p^3, solving an open problem of Saltman.

* Research supported in part by NSF Grant DMS-9100148.
Received June 13, 1995 and in revised form November 21, 1995

Introduction

Let F be a field. In this paper, an "F-division algebra" is a division ring D that is central and finite dimensional over F. Two measures of the size of D are the square root of its F-dimension, or **index**, and its order in the Brauer group, or **period**. Each measure has the same prime factors, and the period always divides the index. However, when they are unequal the index is often a complete mystery, even when the period is obvious from the description of D. This difficulty is at the root of much of the complexity in the theory of division algebras.

Call D **decomposable** if it is F-isomorphic to a tensor product of two nontrivial F-division algebras, or equivalently, if it has a proper nontrivial F-division subalgebra. It is easy to show that when $\mathbf{o}(D) = \mathbf{i}(D)$, the prime-power factors in the "Sylow decomposition" of D are indecomposable. When $\mathbf{o}(D) \neq \mathbf{i}(D)$, the issue is more subtle: Albert found decomposable division algebras of 2-power index in the early 1930's ([Al]), but then indecomposable division algebras of unequal period and index were discovered in 1979, in [ART] and then [Sa1]. It is important to realize that before these latter results, for all anyone knew an F-division algebra had index greater than its period "because" it had a decomposable Sylow factor, in the same way a finite abelian group has order greater than its exponent "because" it has a noncyclic Sylow subgroup. This would have provided an explanation for bad index behavior in terms of the basic structure theory of division algebras. As it stands, the causes of bad index behavior and their relationships to decomposability are not completely understood, and form an interesting area of study.

Over a number field k (i.e., a finite extension of $\mathbb{Q}$), division algebras have equal period and index, and so their decomposability is uninteresting. Over the function field of a nonsingular point on a curve over $\mathbb{Q}$, division algebras of unequal period and index appear, but the Brauer group is so complicated that no one knows how to analyze their decomposability. However, over the function field F of a **Henselian neighborhood** of the point, there remain division algebras of unequal period and index, and their decomposability is both interesting and tractable. The tractability is due to the fact that as far as division algebras are concerned, k and F are not so different (1.0.1). Thus in this paper we work over a **rank one discretely Henselian F with residue field k a number field**. For example, F could be the field of formal power series $k((t))$. Many of the results are proved in the slightly more general setting where k is **absolutely stable**, and the division algebras are **tame**.

We determine exactly how and when given F-division algebras decompose, and mostly determine the answer to the reciprocal problem of how and when they embed properly into other F-division algebras. It turns out that the decomposability of a F-division algebra D of p-power index is determined completely by the behavior of the index when the Brauer class of D is multiplied by p. We show that when $\mathbf{i}(D) \neq \mathbf{o}(D)$, even though D may not itself be decomposable, the division algebra $D^{\otimes n}$ is decomposable for some n (Corollary 3.7). This provides a type of explanation for the erratic index behavior observed when the center is F (Remark 3.7.1).

Some of our results carry over to certain division algebras with centers the rational function field $k(t)$. As a demonstration, we present examples of indecomposable division algebras over $k(t)$ of period p^2 and index p^3 (Corollary 6.4), solving an open problem posed by Saltman (Problem #7, [Sa2]).

Results: Let F be a rank one discretely Henselian field with residue field a number field k. Let D/F be a division algebra. The F-field extensions that do not originate from k are the **totally ramified** (TR) extensions, and the F-division algebras that do not originate from k are the **semiramified** (SR) ones. With some valuation theory it can be shown that a division algebra is semiramified if and only if it has a totally ramified maximal subfield. On the other hand, a standard construction shows that every totally ramified F-extension embeds maximally in some semiramified F-division algebra (this may fail if k is not a number field). Thus the two often appear together. It is shown here that their association endures with respect to their containment in F-division algebras: D contains a totally ramified F-field extension if and only if it contains a semiramified F-division algebra (Corollary 3.8). Interestingly, however, D may have totally ramified subfields not contained in any semiramified subalgebra (Corollary 3.9).

The semiramified division algebras turn out to be the decisive factors in the matter of decomposability over rank one discretely Henselian F: It is proved that when D has p-power index, it is decomposable if and only if it **properly** contains a semiramified F-division algebra. Furthermore, every decomposition has a semiramified component (Theorem 2.2).

This criterion for decomposability has a formulation that makes sense over any field: If D has p-power index, then it is decomposable if and only if $\mathbf{i}(D^{\otimes p}) \neq \frac{1}{p}\,\mathbf{i}(D)$ (Corollary 3.6). Of course, by group theory, the order of D in the Brauer group obeys the rule $\mathbf{o}(D^{\otimes p}) = \frac{1}{p}\mathbf{o}(D)$. Therefore this result says that D is decomposable exactly when its index does not mimic the behavior of its period with respect to tensor product.

Saltman observed in [Sa1] that over **any** field, $\mathbf{i}(D^{\otimes p}) = \frac{1}{p}\,\mathbf{i}(D)$ implies D is indecomposable. Therefore one might ask whether the converse holds in general, as it does for our F. That the answer is no follows from Tignol's indecomposable examples of prime period not equal to index in [Ti] (see also [Jb]).

The criterion for decomposability obtained here for rank one discretely Henselian fields is both nontrivial and effective. Decomposable and indecomposable examples of unequal p-power index and period are produced, and all decompositions that occur are characterized (Corollary 3.2, Theorem 3.3, and Corollary 5.2). It is shown that if D has one division subalgebra, then it has infinitely many nonisomorphic ones, ranging over all indexes dividing the index of D (Corollary 3.10). Curiously, there is sometimes a **gap** in the range of indexes of the semiramified subalgebras (Remark 3.5.3). All proper nontrivial F-division subalgebras are shown to be **maximal** in the (usual) sense that they are not properly contained in any other proper subalgebras of D (Theorem 2.2). In other words, there are at most two nontrivial factors in any decomposition.

Interestingly, the criterion is geometric, in that it involves only dimensional considerations of the division algebra in question ($\mathbf{i}(D^{\otimes p})$ and $\mathbf{i}(D)$), not the specific arithmetic features of the center.

Reciprocal to the problem of decomposability is that of embeddability: If F is discretely Henselian of rank one, and D/F is a division algebra of p-power index, does it embed properly in another F-division algebra that is also of p-power index? Since all proper F-division subalgebras are maximal, this is definitely not true if D is already decomposable.

It is shown here that all semiramified F-division algebras of p-power index do properly embed in F-division algebras of p-power index. There are some restrictions on the invariants of the larger division algebra, as dictated by Theorem 3.3 (Theorem 4.1).

More generally, it is proved that every indecomposable D properly embeds in an F-division algebra of p-power index if and only if for every prime $\mathfrak{p}$ of k, every character over the local field $k_{\mathfrak{p}}$ is the restriction of a character over k of equal order (Theorem 4.3). Thus every F-division algebra properly embeds if k has no Special Case (e.g., contains $\sqrt{-1}$). A particular indecomposable D embeds if the character of D is not "a manifestation" of a Wang counterexample at certain primes in the k-ramification locus of the residue algebra of D. Thus every division algebra of odd p-power index properly embeds. However, if $k = \mathbb{Q}$, it is shown (Corollary 4.4) that there exists an indecomposable division algebra of 2-power index that does not properly embed in any division algebra of 2-power index.

All of this shows that in contrast to the criterion for decomposability, an embeddability criterion would be arithmetic in nature, since it would depend on the arithmetic of k.

The paper concludes with some examples of decompositions (Section 5), and then a discussion of how all of these results apply over the rational function field $k(t)$ (Section 6).

Many of these results hold for any field k such that all tame division algebras over all finite field extensions of k have equal period and index. Such a field is said to be (tamely) **absolutely stable**. By restricting to the particular example of k a number field, existence theorems may be proved, using constructions afforded by class field theory. Thus, with k absolutely stable it is possible to give some properties of any decompositions of D that occur (Theorem 2.2). With k a number field, it is proved that this result is best possible (Theorem 3.3 and Theorem 4.3). With k absolutely stable it is proved that if D is decomposable of p-power index, then p divides a certain invariant of D (Lemma 2.1). With k a number field, the converse is proved (Theorem 3.3), and both decomposable and indecomposable F-division algebras of unequal prime-power period and index are shown to exist.

For other work on decomposability (in addition to that listed above), see [Jb], [Ro], [Sn1], and [SvdB].

The author thanks David Saltman for suggesting that the criterion involving primes established in Corollary 3.6(vi) could be cast into a more general setting (not involving primes), and the University of Texas at Austin for its hospitality during part of the preparation of this paper. He also thanks the referee for motivating improvements in the paper's readability.

1. Background and notation

Let F be a field, and D an F-division algebra. In general, the tensor product of two F-division algebras is not another F-division algebra, but a **central simple** F-algebra, that is, a simple ring central and finite dimensional over F. By Wedderburn's famous theorem, every central simple F-algebra A is a matrix algebra over an F-division algebra. Let $\Delta(A)$ denote this division algebra. Two central simple algebras are **similar** if their underlying division algebras are F-isomorphic. The **Brauer group** $\mathrm{Br}(F)$ is defined to be the set of these similarity classes, with multiplication defined by F-tensor product, and identity element the class of F. Let $[D]$ stand for the class of D in $\mathrm{Br}(F)$. If $\delta \in \mathrm{Br}(F)$, let $\Delta(\delta)$ denote the unique (up to F-isomorphism) F-division algebra representing δ. If p is a prime number, let $\mathrm{Br}(F)(p)$ denote the p-primary part of $\mathrm{Br}(F)$, and if $(p, \mathrm{char}(F)) = 1$, call the elements of $\mathrm{Br}(F)(p)$ **tame**.

If L is an F-field extension, let $\cdot^L$ stand for the algebra tensor product $\cdot \otimes_F L$, which extends scalars from F to L. In general, D^L is not an L-division algebra, but a central simple L-algebra. Let $\cdot^L$ also denote the induced map $\mathrm{Br}(F) \to \mathrm{Br}(L)$ on Brauer elements, called the restriction homomorphism. Let $\mathrm{Br}(L/F)$ be its kernel. Say L **splits** D if $[D] \in \mathrm{Br}(L/F)$, that is, $[D]^L = 0$.

The **index** $\mathbf{i}(D)$ is the square root of the F-dimension of D. This is equal to the smallest degree of all the splitting fields of D. The index of a Brauer element is the index of the representing division algebra. Let $D^{\otimes n}$ denote the division algebra underlying the n-fold tensor product $D \otimes \cdots \otimes D$, that is, the division algebra $\Delta(n[D])$. The **period** $\mathbf{o}(D)$ is the order of $[D]$ in the Brauer group. That is, the smallest n such that $[D^{\otimes n}] \equiv n[D] = 0$. Always $\mathbf{o}(D) \mid \mathbf{i}(D)$, and every prime dividing $\mathbf{i}(D)$ also divides $\mathbf{o}(D)$. Say F is **stable** if $\mathbf{o}(D) = \mathbf{i}(D)$ for all F-division algebras D, and **absolutely stable** if every finite field extension of F is stable. For example, number fields are absolutely stable.

The **character group** $\mathbf{X}(F)$ is the group of of continuous homomorphisms from the absolute Galois group $\mathrm{Gal}(F_{\mathrm{sep}}/F)$ to $\mathbb{Q}/\mathbb{Z}$. If $\chi \in \mathbf{X}(F)$, let $K\langle\chi\rangle$ denote the (cyclic) field extension of degree $|\chi|$ defined by χ. Let $\mathbf{X}(F)(p)$ denote the p-primary part of $\mathbf{X}(F)$. Let $\mathrm{N}(L/F)$ denote the image of the usual norm map from L to F, and set $\mathrm{N}(\chi) = \mathrm{N}(K\langle\chi\rangle/F)$.

For additional references, consult [AT], [JW], [Re], and [Se].

THE SET-UP. Let F be a discretely Henselian-valued field of rank one with absolutely stable residue field k, that is, a field with a discrete valuation of rank one that satisfies Hensel's Lemma (these are called **relatively complete** fields in [S]). If F is complete and contains k, then F is isomorphic to the field of formal power series $k((t))$ in an indeterminate t, with coefficients in k ([Se], Ch. II, §4). Let p be a prime such that $(p, \mathrm{char}(k)) = 1$. By Witt's Theorem ([Se], Ch. XII) there is a split exact sequence

$$(1.0.1) \qquad 0 \longrightarrow \mathrm{Br}(k)(p) \longrightarrow \mathrm{Br}(F)(p) \rightleftarrows \mathbf{X}(k)(p) \longrightarrow 0.$$

The splitting depends on the choice of a uniformizer for F (see below).

Let t be a uniformizer for F. Let $(-,-)$ denote the bihomomorphism

$$\begin{aligned}(-,-)\ :\ \mathbf{X}(F) \times F^{\cdot} &\longrightarrow \mathrm{Br}(F)\\ \chi, s &\longmapsto (\chi, s)\end{aligned}$$

where (χ, s) denotes the Brauer element associated to the usual cyclic crossed product over F defined by χ and s ([Se], Ch. XIV).

With this definition, $(-,t)|_{\mathbf{X}(k)(p)} : \mathbf{X}(k)(p) \longrightarrow \mathrm{Br}(F)(p)$ is the splitting of (1.0.1) that corresponds to the uniformizer t of F, where here $\mathbf{X}(k)(p)$ is identified with the group of **unramified characters** $\mathbf{X}(\mathrm{Gal}(k_{\mathrm{sep}} \otimes_k F/F))(p)$, which is the image of $\mathbf{X}(k)(p)$ in $\mathbf{X}(F)(p)$. If $\delta \in \mathrm{Br}(F)(p)$, then write $\delta = \alpha \dotplus (\chi, t)$, where α is in the image of $\mathrm{Br}(k)(p)$ and χ is in the image of $\mathbf{X}(k)(p)$ under (1.0.1). The element α is called the **inertial** or **unramified** part of δ, and χ is the **character associated to** δ. Occasionally, α will be identified with its preimage in $\mathrm{Br}(k)(p)$, just as χ will be identified with its preimage in $\mathbf{X}(k)(p)$.

Suppose $\delta = \alpha \dotplus (\chi, t)$. The group $\mathrm{Br}(F)$ is abelian, and since α and (χ, t) are in distinct direct summands,

$$(1.0.2) \qquad \mathbf{o}(\delta) = lcm\{\, \mathbf{o}(\alpha), \mathbf{o}((\chi, t)) \,\}.$$

If s is another choice of uniformizer for $F^{\cdot}$ then $s = u \cdot t \cdot x$ for some $u \in k^{\cdot}$ and $x \in U_F^1$, the group of principal units ([Se], Ch. XII). It can be shown using Hensel's Lemma that U_F^1 is divisible, so $U_F^1 \subset \mathrm{N}(\chi)$. Therefore $(\chi, x) = 0$, $(\chi, s) = (\chi, ut)$, and $\delta = (\alpha + (\chi, 1/u)) \dotplus (\chi, ut)$ is the splitting with respect to s. Thus,

$$(1.0.3) \qquad \delta = (\alpha + (\chi, 1/u)) \dotplus (\chi, ut), \quad \forall\, u \in k^{\cdot},$$

and every splitting of (1.0.1), i.e., every choice of uniformizer for F, results in such a splitting of δ, for some $u \in k^{\cdot}$. All of this shows the unramified part of δ depends on t, and is unique (only) $\mathrm{mod}\mathrm{Br}(K\langle\chi\rangle/F)$.

Since the choice of t is arbitrary, the analyses of an element δ and an element $\delta + (\chi, u)$, $u \in k^{\cdot}$, will always be parallel. Call $\delta + (\chi, u)$ a **twist** of δ by u with respect to χ.

Since the valuation on F is Henselian, it extends uniquely to $\Delta(\delta)/F$, and so one may associate to δ the usual valuation theoretic objects ([JW]). Accordingly, call the invariant $\bar{\delta} := \alpha^{K\langle\chi\rangle}$ of δ the **residue element**. Let $\mathbf{f}(\delta)$ denote the **residue index** $\mathbf{i}(\bar{\delta}) = \mathbf{i}(\alpha^{K\langle\chi\rangle})$. Let $\mathbf{e}(\delta)$ denote the **ramification index** $|\chi|$ of the invariant χ, which is also the index (and period) of (χ, t). Note that δ and the twist $\delta + (\chi, u)$ have the same invariants. Call an element δ **unramified** whenever $\mathbf{e}(\delta) = 1$, and **semiramified** (SR) whenever $\mathbf{f}(\delta) = 1$. Thus δ unramified means $\delta = \alpha^F$, for some $\alpha \in \mathrm{Br}(k)$, and δ semiramified means $\delta = (\chi, ut)$ for some unramified $\chi \in \mathbf{X}(F)$ and some $u \in k^{\cdot}$. Of course, these definitions are derived from the valuation theory of the underlying F-division algebras.

The following was first proved by Nakayama ([Na]).

INDEX FORMULA 1.1: *Let $\delta \in \mathrm{Br}(F)$ be tame. Then*

$$\mathbf{i}(\delta) = \mathbf{e}(\delta)\mathbf{f}(\delta).$$

A proof for the complete case appears in [Br1] (Lemma 4). For a proof for general Henselian fields, see [JW] (Theorem 5.15).

The structure of finite field extensions of complete rank one discretely valued F is outlined in [Jn], Chapter II, §5, and the same proofs apply to the Henselian case (see also [S]). If a finite extension is obtained from k by F-scalar extension, then it called **unramified**. It is called **totally ramified** (TR) if it contains no nontrivial unramified subextension. The totally ramified extensions are obtained from F by adjoining a root of ut, for some $u \in k^{\cdot}$. A finite extension E of F consists of its maximal unramified subextension T/F of degree $f(E/F)$, called the **residue degree** of E/F, followed by a totally ramified extension E/T of degree $e(E/F)$, called the **ramification index** of E/F.

In [Br2], the totally ramified extensions and their relationships to F-division algebras are analyzed, and the following definitions are made:

Definition 1.2: Let D/F be central simple, with Brauer class δ. The **ceiling number** $c_u(D,t) \equiv c_u(\delta,t)$ for $u \in k^{\cdot}$ is the largest n such that $F(\sqrt[n]{ut}) \subset \Delta(D)$. The **upper ceiling number** $\bar{\mathbf{c}}(D) \equiv \bar{\mathbf{c}}(\delta)$ and the **lower ceiling number** $\underline{\mathbf{c}}(D) \equiv \underline{\mathbf{c}}(\delta)$ are the least common multiple and the greatest common divisor of the $c_u(D,t)$, respectively, taken over all $u \in k^{\cdot}$. Call D or δ **stubbed** if $\bar{\mathbf{c}}(D) \mid \mathbf{f}(D)$, and **stepped** if $\bar{\mathbf{c}}(D) \nmid \mathbf{f}(D)$.

By this definition, $\bar{\mathbf{c}}(D)$ is the degree over F of the largest totally ramified subfield of $\Delta(D)$, and $\underline{\mathbf{c}}(D)$ is the number n such that $\Delta(D)$ contains all totally ramified extensions of degree dividing n.

The following is a summary of results that appear in [Br2].

THEOREM 1.3: *Suppose $\delta = \alpha \dotplus (\chi, t) \in \mathrm{Br}(F)$ is tame, as per (1.0.1). Then*

(i) $c_u(\delta,t), \underline{\mathbf{c}}(\delta)$, *and* $\bar{\mathbf{c}}(\delta)$ *all divide* $\mathbf{e}(\delta)$.

(ii) $c_u(\delta,t) = \sup_{n \mid \mathbf{e}(\delta)} \{n \mid \mathbf{i}((\alpha + (\chi, 1/u))^{K\langle n\chi\rangle}) = \mathbf{f}(\delta)\}$.

(iii) $n \mid \underline{\mathbf{c}}(\delta) \Longleftrightarrow n \mid \mathbf{e}(\delta)$ *and* $\mathbf{i}((\alpha + (\chi, u))^{K\langle n\chi\rangle}) = \mathbf{f}(\delta) \quad \forall\, u \in k^{\cdot}$.

(iv) $\underline{\mathbf{c}}(\delta)$ *and* $\bar{\mathbf{c}}(\delta)$ *are invariants of* δ, *and are the same for all twists* $\delta + (\chi, u)$ *of* δ *with respect to* χ.

(v) *If* δ *is stubbed, then* $\underline{\mathbf{c}}(\delta) = \bar{\mathbf{c}}(\delta) \mid \mathbf{f}(\delta)$, *and* $\Delta(\delta)$ *contains all totally ramified extensions of degree dividing* $\bar{\mathbf{c}}(\delta)$ *(but none of larger degree).*

(vi) *If* k *is a number field, then always* $\underline{\mathbf{c}}(\delta) \mid \mathbf{f}(\delta)$, *and* δ *is stepped if and only if* $\underline{\mathbf{c}}(\delta) = \mathbf{f}(\delta)$, $\mathbf{f}(\delta) \neq \mathbf{e}(\delta)$, *and* $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$.

DECOMPOSABILITY. Call an F-division algebra D **decomposable** if it can be written as a tensor product of 2 nontrivial F-division algebras, i.e., $D \cong D_1 \otimes D_2$ for nontrivial F-division algebras D_1 and D_2. A dimension count forces $\mathbf{i}(D) = \mathbf{i}(D_1)\,\mathbf{i}(D_2)$. Call D **indecomposable** if it isn't decomposable.

D is decomposable if and only if D properly contains a nontrivial F-division algebra: For both of the D_i embed in D as subalgebras. Conversely, one can use the Double Centralizer Theorem ([Re] (7.11), (7.13)) to show that any nontrivial proper F-division subalgebra D_1 tensored with its centralizer D_2 is isomorphic to D, hence a decomposition of D.

We will usually prefer to deal in Brauer elements. If the Brauer class of D is δ, then it is easy to see that D is decomposable if and only if there exist nontrivial Brauer elements δ_1 and δ_2 such that $\delta = \delta_1 + \delta_2$ and $\mathbf{i}(\delta) = \mathbf{i}(\delta_1)\,\mathbf{i}(\delta_2)$. Call an element $\delta \in \mathrm{Br}(F)$ **multiplicatively decomposable** if its representing division algebra $D = \Delta(\delta)$ is decomposable, and **indecomposable** if not. In general,

$$\delta = \delta_1 + \delta_2 \implies \mathbf{i}(\delta) \mid \mathbf{i}(\delta_1) \cdot \mathbf{i}(\delta_2). \tag{1.3.1}$$

Since an element δ always has a "Sylow decomposition" into factors of prime power index, **when discussing decomposability, always assume that** $\mathbf{i}(\delta)$

is a prime power not divisible by $\mathrm{char}(k)$. If $\mathbf{i}(\delta)$ is a prime power and $\mathbf{i}(\delta) = \mathbf{o}(\delta)$, then it is easy to show that $\Delta(\delta)$ is indecomposable. Call $\Delta(\delta)$ **trivially indecomposable** in this case. Say $\Delta(\delta)$ has a (nontrivial) **split decomposition** if it has a decomposition into an unramified factor and a semiramified factor. Then for some choice of t, the expression $\Delta(\delta) \cong \Delta(\alpha) \otimes \Delta(\chi, t)$ **is already a decomposition.**

LEMMA 1.4: *Let p be prime, with $(p, \mathrm{char}(k)) = 1$. Suppose $\delta = \alpha \dotplus (\chi, t) \in \mathrm{Br}(F)(p)$. Then*

$$\Delta(\delta) \textit{ has a split decomposition} \iff \mathbf{f}(\delta) \neq 1 \textit{ and } \bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$$

$$\Delta(\delta) \textit{ is trivially indecomposable} \iff \delta \textit{ is semiramified or } K\langle\chi\rangle \subset \Delta(\alpha).$$

Proof: Suppose $\Delta(\delta)$ has the split decomposition $\Delta(\delta) \cong \Delta(\alpha) \otimes \Delta(\chi, t)$. Then $\mathbf{i}(\delta) = \mathbf{i}(\alpha)\,\mathbf{i}((\chi, t))$, hence $\mathbf{i}(\alpha) = \mathbf{f}(\delta) \neq 1$ by Index Formula 1.1. Since D visibly has a TR subfield of degree $\mathbf{e}(D)$, namely $F(\sqrt[\mathbf{e}(D)]{t}) \subset \Delta(\chi, t)$, $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$ by Definition 1.2. Conversely, if $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$, then for some $u \in k^{\cdot}$, $\mathbf{f}(\delta) \equiv \mathbf{i}((\alpha + (\chi, 1/u))^{K\langle\chi\rangle}) = \mathbf{i}(\alpha + (\chi, 1/u))$ by Theorem 1.3(iii), so that $\mathbf{i}(\delta) = \mathbf{i}(\alpha + (\chi, 1/u)) \cdot \mathbf{i}((\chi, ut))$. If $\mathbf{f}(\delta) \neq 1$, then this shows $\Delta(\delta) \cong \Delta(\alpha + (\chi, 1/u)) \otimes \Delta(\chi, ut)$ is a split decomposition, as desired.

If δ has prime-power index and $\mathbf{i}(\delta) = \mathbf{o}(\delta)$ (so $\Delta(\delta)$ is trivially indecomposable), then since $\mathbf{i}(\delta) = \mathbf{i}(\alpha^{K\langle\chi\rangle}) \cdot \mathbf{i}((\chi, t))$ (by Index Formula 1.1) and $\mathbf{o}(\delta) = \mathit{lcm}\{\mathbf{o}(\alpha), \mathbf{o}((\chi, t))\}$ (by 1.0.2), either $\mathbf{i}(\alpha^{K\langle\chi\rangle}) \cdot \mathbf{i}((\chi, t)) = \mathbf{o}((\chi, t))$, or $\mathbf{i}(\alpha^{K\langle\chi\rangle}) \cdot \mathbf{i}((\chi, t)) = \mathbf{o}(\alpha)$. In the first case, since $\mathbf{i}((\chi, t)) = \mathbf{o}((\chi, t))$, $\mathbf{i}(\alpha^{K\langle\chi\rangle}) \equiv \mathbf{f}(\delta)$ must be trivial, and δ must be SR. In the second case, since $\mathbf{i}((\chi, t)) = [K\langle\chi\rangle : F]$ and $\mathbf{o}(\alpha) = \mathbf{i}(\alpha)$, necessarily $[K\langle\chi\rangle : F]\,\mathbf{i}(\alpha^{K\langle\chi\rangle}) = \mathbf{i}(\alpha)$, hence $K\langle\chi\rangle \subset \Delta(\alpha)$ by the basic theory. Conversely, it is easy to see that if δ is SR or $K\langle\chi\rangle \subset \Delta(\alpha)$, then $\Delta(\delta)$ is trivially indecomposable. ∎

As mentioned previously, all results must be essentially indifferent to the replacement of each δ by $\delta + (\chi, u)$, $u \in k^{\cdot}$, where χ is the character of δ. Indeed, let χ_i be the character of the element δ_i $(i = 1, 2)$. If $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta_2)$ is a decomposition, then $\Delta(\delta + (\chi, u)) \cong \Delta(\delta_1 + (\chi_1, u)) \otimes \Delta(\delta_2 + (\chi_2, u))$: For the addition is correct since $(\chi, u) = (\chi_1, u) + (\chi_2, u)$, and the indexes are multiplicative since $\mathbf{i}(\delta + (\chi, u)) = \mathbf{i}(\delta)$ and $\mathbf{i}(\delta_i + (\chi_i, u)) = \mathbf{i}(\delta_i)$.

THE SPECIAL CASE. In Section 4 the so-called **Special Case** of the Grunwald–Wang Theorem, which is relevant to number fields k, will be applied. Recall the definition ([AT], Ch.X): Let k be a number field, and let r and s be numbers maximal such that the primitive root of unity $\zeta_{2^r} \in k^{\cdot}$, and $\zeta_{2^s} \in k(\zeta_{2^{r+1}})^{\cdot}$. Let $S_0 = \{\mathfrak{p} \mid k_{\mathfrak{p}}(\zeta_{2^{s+1}})/k_{\mathfrak{p}}$ is noncyclic$\}$. Then say k has a **Special Case** if $k(\zeta_{2^{s+1}})/k$ is noncyclic, and a **nonempty Special Case** if S_0 is nonempty. For example, it can be shown ([AT], Ch. X) that $\mathbb{Q}$ has a (nonempty) Special Case, but any field containing $\sqrt{-1}$ does not. The Special Case is an obstacle to the lifting of a finite family $\{\psi_{\mathfrak{p}}\}$ (distinct $\mathfrak{p}$) of local characters to a global character ψ when $2^{s+1} \mid m = lcm\{\,|\psi_{\mathfrak{p}}|\,\}$, $S_0 \neq \varnothing$, S_0 is contained in the (finite) set of primes $\mathfrak{p}$, and $\sum_{S_0} \psi_{\mathfrak{p}}((\zeta_{2^s}+1)^m) \neq 0$. Then, any $\psi \in \mathbf{X}(k)$ that restricts to the $\psi_{\mathfrak{p}}$ has order divisible by $2m$, and the set $\{\psi_{\mathfrak{p}}\}$ constitutes a **Wang counterexample**. Otherwise, there exists such a ψ of any order divisible by m, and the set $\{\psi_{\mathfrak{p}}\}$ has a **Grunwald lift**. Note that the Special Case is irrelevant when working with odd numbers m.

2. Decomposability

As in the last section, let F be a rank one discretely Henselian field with absolutely stable residue field k, and let p be a prime of $\mathbb{Q}$ such that $(p, \mathrm{char}(k)) = 1$.

Suppose $\Delta(\delta)/F$ is decomposable (of p-power index), and $\Delta(\delta) \cong \Delta(\delta_1)\otimes\Delta(\delta_2)$ is a decomposition. Set $\delta = \alpha \dotplus (\chi, t)$ and $\delta_i = \alpha_i \dotplus (\chi_i, t)$. Since $\mathbf{i}(\delta) = \mathbf{i}(\delta_1)\,\mathbf{i}(\delta_2)$,

$$\mathbf{e}(\delta)\mathbf{f}(\delta) = \mathbf{e}(\delta_1)\mathbf{f}(\delta_1)\mathbf{e}(\delta_2)\mathbf{f}(\delta_2) \tag{2.0.1}$$

by Index Formula 1.1.

The next lemma is crucial in all that follows.

LEMMA 2.1: *Suppose $\Delta(\delta)$ is decomposable and has p-power index, as in the above set-up. Let $\epsilon = \min_i\{\mathbf{o}(\delta_i)\}$. Then*

$$\epsilon \mid (\mathbf{f}(\delta), \underline{\mathbf{c}}(\delta)).$$

In particular, $p \mid \underline{\mathbf{c}}(\delta)$. Moreover:

(i) *Any nontrivial decomposition of $\Delta(\delta)$ has the form $\Delta(\delta) \cong \Delta(\delta_1)\otimes\Delta(\delta_2)$, such that $\mathbf{e}(\delta_1) \mid (\underline{\mathbf{c}}(\delta), \mathbf{e}(\delta_2))$, and $\mathbf{e}(\delta_2) = \mathbf{e}(\delta)$.*

(ii) *For each $u \in k^{\cdot}$, $\mathbf{i}(\alpha_i + (\chi_i, u)) \mid \mathbf{i}(\alpha + (\chi, u))$ for each i.*

(iii) *If $\underline{\mathbf{c}}(\delta) \neq \mathbf{e}(\delta)$, then $\epsilon < (\mathbf{i}(\alpha + (\chi, u)), \mathbf{e}(\delta))$ for some $u \in k^{\cdot}$.*

(iv) *If* $\bar{\mathbf{c}}(\delta) \neq \mathbf{e}(\delta)$, *then* $\epsilon < (\mathbf{i}(\alpha + (\chi, u)), \mathbf{e}(\delta))$ *for all* $u \in k^{\cdot}$.

Proof: Albert showed that over any field,

$$p \mid \mathbf{i}(\delta) \implies \mathbf{i}(p\delta) \mid \frac{1}{p}\mathbf{i}(\delta), \tag{2.1.1}$$

and Saltman observed that equality implies $\Delta(\delta)$ is indecomposable ([Sal]).

CLAIM:

$$\mathbf{i}(n\delta) = \frac{1}{n^2}\mathbf{i}(\delta) \iff n \mid (\mathbf{f}(\delta), \underline{\mathbf{c}}(\delta)). \tag{2.1.2}$$

By Index Formula 1.1, $\mathbf{i}(n\delta) = \mathbf{e}(n\delta)\mathbf{f}(n\delta)$. Since $n\delta = n\alpha \dot{+} (n\chi, t)$, both $\mathbf{e}(n\delta) \equiv |n\chi| = \mathbf{e}(\delta)/(\mathbf{e}(\delta), n)$, and $\mathbf{f}(n\delta) = \mathbf{i}(n\alpha^{K\langle n\chi\rangle})$. Of course, $\mathbf{e}(\delta)/(\mathbf{e}(\delta), n) \geq \frac{1}{n}\mathbf{e}(\delta)$. Since k is absolutely stable, $\mathbf{i}(n\alpha^{K\langle n\chi\rangle}) \geq \frac{1}{n}\mathbf{i}(\alpha^{K\langle n\chi\rangle})$, and always $\mathbf{i}(\alpha^{K\langle n\chi\rangle}) \geq \mathbf{i}(\alpha^{K\langle \chi\rangle}) \equiv \mathbf{f}(\delta)$, by the basic theory. (Here the unramified elements α and χ are identified with their preimages over k.) Hence $\mathbf{f}(n\delta) \geq \frac{1}{n}\mathbf{f}(\delta)$. Thus multiplication by n reduces the ramification and residue indexes each by at most n, and

$$\mathbf{i}(n\delta) \geq \frac{1}{n^2}\mathbf{e}(\delta)\mathbf{f}(\delta) = \frac{1}{n^2}\mathbf{i}(\delta) \ \forall n, \tag{2.1.3}$$

with equality if and only if $\mathbf{e}(n\delta) = \frac{1}{n}\mathbf{e}(\delta)$ and $\mathbf{f}(n\delta) = \frac{1}{n}\mathbf{f}(\delta)$. To prove the claim it now suffices to show $\mathbf{e}(n\delta) = \frac{1}{n}\mathbf{e}(\delta)$ and $\mathbf{f}(n\delta) = \frac{1}{n}\mathbf{f}(\delta)$ if and only if $n \mid (\mathbf{f}(\delta), \underline{\mathbf{c}}(\delta))$.

Since $\mathbf{f}(n\delta) = \mathbf{i}(n\alpha^{K\langle n\chi\rangle})$, $\mathbf{f}(\delta) = \mathbf{i}(\alpha^{K\langle\chi\rangle})$, and k is absolutely stable, $\mathbf{f}(n\delta) = \frac{1}{n}\mathbf{f}(\delta)$ if and only if both n divides $\mathbf{i}(\alpha^{K\langle n\chi\rangle})$, and $\mathbf{i}(\alpha^{K\langle n\chi\rangle})$ is equal to $\mathbf{i}(\alpha^{K\langle\chi\rangle})$. This is equivalent to $\max\{1, \mathbf{i}(\alpha^{K\langle n\chi\rangle})/n\} = \frac{1}{n}\mathbf{f}(\delta)$, i.e.,

$$\max\{n, \mathbf{i}(\alpha^{K\langle n\chi\rangle})\} = \mathbf{f}(\delta).$$

For if $n \mid \mathbf{i}(\alpha^{K\langle n\chi\rangle})$, then $\max\{1, \mathbf{i}(\alpha^{K\langle n\chi\rangle})/n\} = \mathbf{i}(\alpha^{K\langle n\chi\rangle})/n$, and if in addition $\mathbf{i}(\alpha^{K\langle n\chi\rangle}) = \mathbf{f}(\delta)$, then $\max\{1, \mathbf{i}(\alpha^{K\langle n\chi\rangle})/n\} = \mathbf{f}(\delta)/n$. Conversely, if $\max\{1, \mathbf{i}(\alpha^{K\langle n\chi\rangle})/n\} = \mathbf{f}(\delta)/n$, then since $\mathbf{f}(\delta) \leq \mathbf{i}(\alpha^{K\langle n\chi\rangle})$,

$$\max\{1, \mathbf{i}(\alpha^{K\langle n\chi\rangle})/n\} = \mathbf{i}(\alpha^{K\langle n\chi\rangle})/n,$$

hence $n \mid \mathbf{i}(\alpha^{K\langle n\chi\rangle})$, and $\mathbf{f}(\delta) = \mathbf{i}(\alpha^{K\langle n\chi\rangle})$.

Always $\mathbf{i}(\alpha^{K\langle n\chi\rangle}) \geq \mathbf{i}(\alpha^{K\langle\chi\rangle}) \equiv \mathbf{f}(\delta)$, so $\max\{n, \mathbf{i}(\alpha^{K\langle n\chi\rangle})\} = \mathbf{f}(\delta)$ if and only if $n \mid \mathbf{f}(\delta)$ and $\mathbf{i}(\alpha^{K\langle n\chi\rangle}) = \mathbf{f}(\delta)$. By Theorem 1.3(iii), this is equivalent to $n \mid \mathbf{f}(\delta)$ and $n \mid \underline{\mathbf{c}}(\delta)$. Therefore $\mathbf{f}(n\delta) = \frac{1}{n}\mathbf{f}(\delta)$ if and only if $n \mid (\mathbf{f}(\delta), \underline{\mathbf{c}}(\delta))$. Since $\underline{\mathbf{c}}(\delta) \mid \mathbf{e}(\delta)$ (by Theorem 1.3(i)), $n \mid (\mathbf{f}(\delta), \underline{\mathbf{c}}(\delta))$ implies that $n \mid \mathbf{e}(\delta)$, hence $\mathbf{f}(n\delta) = \frac{1}{n}\mathbf{f}(\delta)$ implies that $\mathbf{e}(n\delta) = \frac{1}{n}\mathbf{e}(\delta)$. This proves the claim.

By definition, the number ϵ divides $\mathbf{o}(\delta_1)$ and $\mathbf{o}(\delta_2)$, hence it divides $\mathbf{i}(\delta_1)$ and $\mathbf{i}(\delta_2)$. Therefore by (2.1.1), $\mathbf{i}(\epsilon\delta_i) \mid \frac{1}{\epsilon}\,\mathbf{i}(\delta_i)$. Since $\epsilon\delta = \epsilon\delta_1 + \epsilon\delta_2$, $\mathbf{i}(\epsilon\delta) \mid \mathbf{i}(\epsilon\delta_1)\,\mathbf{i}(\epsilon\delta_2)$ by (1.3.1), hence $\mathbf{i}(\epsilon\delta) \mid \frac{1}{\epsilon^2}\,\mathbf{i}(\delta_1)\,\mathbf{i}(\delta_2) = \frac{1}{\epsilon^2}\,\mathbf{i}(\delta)$. By (2.1.3), $\mathbf{i}(\epsilon\delta) \geq \frac{1}{\epsilon^2}\,\mathbf{i}(\delta)$, hence $\mathbf{i}(\epsilon\delta) = \frac{1}{\epsilon^2}\,\mathbf{i}(\delta)$. Therefore, by the claim, $\epsilon \mid (\mathbf{f}(\delta), \underline{\mathbf{c}}(\delta))$, proving the first part of the lemma.

Since each $\mathbf{e}(\delta_i) \mid \mathbf{o}(\delta_i)$ by (1.0.2), and one of the $\mathbf{o}(\delta_i)$ equals ϵ, $\mathbf{e}(\delta_i) \mid \epsilon$ for some i. Since $\Delta(\delta)$ is decomposable, $\epsilon \mid \underline{\mathbf{c}}(\delta)$ by the above, and by Theorem 1.3(i), $\underline{\mathbf{c}}(\delta) \mid \mathbf{e}(\delta)$. Conclude $\mathbf{e}(\delta_i) \mid \mathbf{e}(\delta)$ for some i. Since $\chi = \chi_1 + \chi_2$, abelian group theory then implies that $\mathbf{e}(\delta_1)$ and $\mathbf{e}(\delta_2)$ both divide $\mathbf{e}(\delta)$, and the larger is equal to $\mathbf{e}(\delta)$. Since all elements have p-power order, either $\mathbf{e}(\delta_1) \mid \mathbf{e}(\delta_2)$ or $\mathbf{e}(\delta_2) \mid \mathbf{e}(\delta_1)$. Switch indices if necessary to make $\mathbf{e}(\delta_1) \mid \mathbf{e}(\delta_2)$, so then $\mathbf{e}(\delta_2) = \mathbf{e}(\delta)$. Since $\mathbf{e}(\delta_i) \mid \epsilon$ for some i, $\mathbf{e}(\delta_1) \mid \epsilon \mid \underline{\mathbf{c}}(\delta)$ by the above. This proves (i).

As above, assume $\mathbf{e}(\delta_1) \mid \mathbf{e}(\delta_2)$, so then $\mathbf{e}(\delta_2) = \mathbf{e}(\delta)$, and (2.0.1) becomes

$$\mathbf{f}(\delta) = \mathbf{e}(\delta_1)\mathbf{f}(\delta_1)\mathbf{f}(\delta_2). \tag{2.1.4}$$

By (1.0.2), $\mathbf{o}(\alpha_1) \mid \mathbf{o}(\delta_1)$, and always $\mathbf{o}(\delta_1) \mid \mathbf{i}(\delta_1)$ by the basic theory. Therefore since $\mathbf{i}(\delta_1) \mid \mathbf{f}(\delta)$ by (2.1.4), $\mathbf{o}(\alpha_1) \mid \mathbf{f}(\delta)$. Also, $\mathbf{f}(\delta) \equiv \mathbf{i}(\alpha^{K\langle\chi\rangle}) \mid \mathbf{i}(\alpha)$, and since k is absolutely stable, $\mathbf{f}(\delta) \mid \mathbf{o}(\alpha)$. Putting these together yields $\mathbf{o}(\alpha_1) \mid \mathbf{o}(\alpha)$, hence, since $\alpha = \alpha_1 + \alpha_2$, $\mathbf{o}(\alpha_2) \mid \mathbf{o}(\alpha)$ also (again by abelian group theory). It is easy to show that all of this works if α and α_i are replaced by $\alpha + (\chi, u)$ and $\alpha_i + (\chi_i, u)$, respectively, for any $u \in k^{\cdot}$. This proves (ii), since k is absolutely stable.

If $\underline{\mathbf{c}}(\delta) \neq \mathbf{e}(\delta)$, then $\epsilon < \mathbf{e}(\delta)$ (since $\epsilon \mid \underline{\mathbf{c}}(\delta) \mid \mathbf{e}(\delta)$). Since $\mathbf{e}(\delta) = \mathbf{e}(\delta_2)$ and $\mathbf{e}(\delta_2) = |\chi_2| \leq \mathbf{o}(\delta_2)$ (by (1.0.2)), $\epsilon < \mathbf{e}(\delta) \Rightarrow \epsilon < \mathbf{e}(\delta_2) \leq \mathbf{o}(\delta_2)$ and $\epsilon \equiv \mathbf{o}(\delta_1) < \mathbf{e}(\delta)$. By Definition 1.2 of $\underline{\mathbf{c}}(\delta)$ and Theorem 1.3(iii), $\underline{\mathbf{c}}(\delta) \neq \mathbf{e}(\delta)$ implies $\mathbf{f}(\delta) < \mathbf{i}(\alpha + (\chi, u))$ for some $u \in k^{\cdot}$, and since $\mathbf{i}(\delta_1) = \mathbf{e}(\delta_1)\mathbf{f}(\delta_1) \leq \mathbf{f}(\delta)$, $\mathbf{i}(\delta_1) < \mathbf{i}(\alpha + (\chi, u))$, hence $\mathbf{o}(\delta_1) < \mathbf{i}(\alpha + (\chi, u))$. Therefore $\epsilon < (\mathbf{i}(\alpha + (\chi, u)), \mathbf{e}(\delta))$ for some $u \in k^{\cdot}$. This proves (iii).

If $\bar{\mathbf{c}}(\delta) \neq \mathbf{e}(\delta)$, then of course $\underline{\mathbf{c}}(\delta) \neq \mathbf{e}(\delta)$, but by the definition of $\bar{\mathbf{c}}(\delta)$, this time $\mathbf{o}(\delta_1) < (\mathbf{i}(\alpha + (\chi, u)), \mathbf{e}(\delta))$ for **all** $u \in k^{\cdot}$. This completes the proof. ∎

Remark 2.1.5: The author thanks B.A. Sethuraman for suggesting the use of Saltman's observation on (2.1.1) to improve the proof of this lemma ([Sn2]).

THEOREM 2.2: *Suppose δ has p-power index, and $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta_2)$ is a decomposition, with $\mathbf{e}(\delta_1) \mid \mathbf{e}(\delta_2)$. Then $\Delta(\delta_1)$ and $\Delta(\delta_2)$ are both indecomposable, and one of them is semiramified (SR). Moreover, either*

(i) *δ_1 is SR and δ_2 is not SR, or*

(ii) *$\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$ (i.e., there exists a split decomposition), δ_2 is SR, and $\mathbf{i}(\delta_1) = \mathbf{o}(\delta_1) = \mathbf{f}(\delta)$.*

Proof: By (2.1.3), $\frac{1}{p^2}\,\mathbf{i}(\delta) \mid \mathbf{i}(p\delta)$. Since $p\delta = p\delta_1 + p\delta_2$, $\mathbf{i}(p\delta) \mid \mathbf{i}(p\delta_1)\,\mathbf{i}(p\delta_2)$ by (1.3.1). Each δ_i is nontrivial, hence by (2.1.1), $\mathbf{i}(p\delta_i) \mid \frac{1}{p}\,\mathbf{i}(\delta_i)$. Since $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta_2)$ is a decomposition (by hypothesis), $\mathbf{i}(\delta_1)\,\mathbf{i}(\delta_2) = \mathbf{i}(\delta)$. Putting all of this together yields $\frac{1}{p^2}\,\mathbf{i}(\delta) \mid \mathbf{i}(p\delta) \mid \mathbf{i}(p\delta_1)\,\mathbf{i}(p\delta_2) \mid \frac{1}{p^2}\,\mathbf{i}(\delta_1)\,\mathbf{i}(\delta_2) = \frac{1}{p^2}\,\mathbf{i}(\delta)$. Both ends are equal, and the desired conclusion is that $\mathbf{i}(p\delta_i) = \frac{1}{p}\,\mathbf{i}(\delta_i)$. Therefore $\Delta(\delta_i)$ is indecomposable by Saltman's observation on (2.1.1).

As usual, set $\delta = \alpha \dotplus (\chi, t)$ and $\delta_i = \alpha_i \dotplus (\chi_i, t)$. Since $\mathbf{e}(\delta_1) \mid \mathbf{e}(\delta_2)$ by hypothesis,

$$(2.2.1)\qquad \mathbf{e}(\delta_1) \mid \mathbf{e}(\delta) = \mathbf{e}(\delta_2),\;\; \mathbf{f}(\delta) = \mathbf{e}(\delta_1)\mathbf{f}(\delta_1)\mathbf{f}(\delta_2),\;\; \text{and}\;\; \mathbf{e}(\delta_1) \mid (\mathbf{f}(\delta), \underline{\mathbf{c}}(\delta)).$$

The first two equations follow from (2.1.4) and the argument leading up to it. The third follows from the observation that since $\mathbf{e}(\delta_1) \mid \mathbf{e}(\delta_2)$, $\mathbf{e}(\delta_1)$ must divide $\mathbf{o}(\delta_2)$, hence $\mathbf{e}(\delta_1)$ divides $(\mathbf{o}(\delta_1), \mathbf{o}(\delta_2))$, which divides $(\mathbf{f}(\delta), \underline{\mathbf{c}}(\delta))$ by Lemma 2.1.

Since $\mathbf{e}(\delta_1) \mid \underline{\mathbf{c}}(\delta)$,

$$(2.2.2)\qquad \mathbf{i}(\alpha^{K\langle \mathbf{e}(\delta_1)\chi\rangle}) = \mathbf{f}(\delta)$$

by Theorem 1.3(iii). The technique for the rest of the proof is to examine the result of extending the base to $K\langle \mathbf{e}(\delta_1)\chi\rangle$. To simplify notation, abbreviate $\mathbf{e}(\delta)$ to $\mathbf{e}$, $\mathbf{e}(\delta_i)$ to $\mathbf{e}_i$, and similarly for the other invariants.

CASE 1: $\mathbf{i}(\alpha_2^{K\langle \mathbf{e}_1\chi\rangle}) \geq \mathbf{i}(\alpha^{K\langle \mathbf{e}_1\chi\rangle})$. Since $\chi = \chi_1 + \chi_2$, $\mathbf{e}_1\chi = \mathbf{e}_1\chi_2$, hence $K\langle \mathbf{e}_1\chi\rangle = K\langle \mathbf{e}_1\chi_2\rangle$. Always $\mathbf{e}_1\mathbf{f}_2 \geq [K\langle\chi_2\rangle : K\langle \mathbf{e}_1\chi_2\rangle]\,\mathbf{i}(\alpha_2^{K\langle\chi_2\rangle}) \geq \mathbf{i}(\alpha_2^{K\langle \mathbf{e}_1\chi_2\rangle})$, and since $K\langle \mathbf{e}_1\chi\rangle = K\langle \mathbf{e}_1\chi_2\rangle$, $\mathbf{e}_1\mathbf{f}_2 \geq \mathbf{i}(\alpha_2^{K\langle \mathbf{e}_1\chi\rangle})$. By hypothesis, $\mathbf{i}(\alpha_2^{K\langle \mathbf{e}_1\chi\rangle}) \geq \mathbf{i}(\alpha^{K\langle \mathbf{e}_1\chi\rangle})$, and by (2.2.2), $\mathbf{i}(\alpha^{K\langle \mathbf{e}_1\chi\rangle}) = \mathbf{f}$. But by (2.2.1), $\mathbf{f} = \mathbf{e}_1\mathbf{f}_1\mathbf{f}_2$! Therefore $\mathbf{e}_1\mathbf{f}_2 \geq \mathbf{e}_1\mathbf{f}_1\mathbf{f}_2$, hence $\mathbf{f}_1 = 1$. Thus δ_1 is SR.

CASE 2: $\mathbf{i}(\alpha_2^{K\langle \mathbf{e}_1\chi\rangle}) < \mathbf{i}(\alpha^{K\langle \mathbf{e}_1\chi\rangle})$. Since k is absolutely stable, this means $\mathbf{o}(\alpha_2^{K\langle \mathbf{e}_1\chi\rangle}) < \mathbf{o}(\alpha^{K\langle \mathbf{e}_1\chi\rangle})$. Therefore since $\alpha^{K\langle \mathbf{e}_1\chi\rangle} = \alpha_1^{K\langle \mathbf{e}_1\chi\rangle} + \alpha_2^{K\langle \mathbf{e}_1\chi\rangle}$, $\mathbf{i}(\alpha^{K\langle \mathbf{e}_1\chi\rangle}) = \mathbf{i}(\alpha_1^{K\langle \mathbf{e}_1\chi\rangle})$. Therefore by (2.2.2), $\mathbf{i}(\alpha_1^{K\langle \mathbf{e}_1\chi\rangle}) = \mathbf{f}$, hence $\mathbf{i}(\alpha_1) \geq \mathbf{f}$. Since $\mathbf{e}_1\mathbf{f}_1 \geq \mathbf{i}(\alpha_1)$, this yields a string of inequalities $\mathbf{i}(\alpha_1) \geq \mathbf{f} = \mathbf{e}_1\mathbf{f}_1\mathbf{f}_2 \geq \mathbf{f}_2\,\mathbf{i}(\alpha_1) \geq \mathbf{i}(\alpha_1)$. Therefore $\mathbf{i}(\alpha_1) = \mathbf{f} = \mathbf{e}_1\mathbf{f}_1$, and $\mathbf{f}_2 = 1$. This shows δ_2 is SR, as contended. Moreover, $\mathbf{i}(\delta_1) = \mathbf{i}(\alpha_1) = \mathbf{f}$. By (1.0.2), $\mathbf{i}(\alpha_1) \equiv \mathbf{o}(\alpha_1) \,|\, \mathbf{o}(\delta_1)$, and always $\mathbf{o}(\delta_1) \,|\, \mathbf{i}(\delta_1)$. Conclude $\mathbf{o}(\delta_1) = \mathbf{i}(\delta_1) = \mathbf{f}$.

It remains to prove (i) and (ii). To prove (i), suppose δ_2 is not SR. Then Case 1 above applies (since δ_2 is SR in Case 2), hence δ_1 is SR. To prove (ii), suppose δ_2 is SR. Then since $\mathbf{e} = \mathbf{e}_2$, there exists a TR subfield of $\Delta(\delta_2)$ of degree $\mathbf{e}$. Since $\Delta(\delta_2) \subset \Delta(\delta)$, the same is true of $\Delta(\delta)$. Therefore (by Def. 1.2) $\bar{\mathbf{c}} = \mathbf{e}$. Since $\mathbf{e} = \mathbf{e}_2$, $\mathbf{f} = \mathbf{i}(\delta_1)$ by (2.2.1), hence $\mathbf{f} \neq 1$. By Lemma 1.4, $\mathbf{f} \neq 1$ and $\bar{\mathbf{c}} = \mathbf{e}$ are together equivalent to existence of a (nontrivial) split decomposition, as contended. In Case 1, δ_1 is SR, so $\mathbf{i}(\delta_1) = \mathbf{o}(\delta_1) = \mathbf{e}_1$, and by (2.2.1), $\mathbf{f} = \mathbf{e}_1$. Conclude $\mathbf{i}(\delta_1) = \mathbf{o}(\delta_1) = \mathbf{f}$, as desired. In Case 2, already it has been demonstrated that $\mathbf{i}(\delta_1) = \mathbf{o}(\delta_1) = \mathbf{f}$. This completes the proof. ∎

Remark 2.2.3: Stronger restrictions on the factors of a decomposition cannot be made: In Theorem 3.3 it will be shown that for a given D, there exist decompositions $D \cong D_1 \otimes D_2$ with $\mathbf{e}(\delta_1) \,|\, \mathbf{e}(\delta_2)$ and $\mathbf{e}(\delta_1)$ any value between 1 and $(\mathbf{f}(\delta), \underline{\mathbf{c}}(\delta))$, which is all of the latitude allowed by Lemma 2.1(i). If $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$, examples as in (ii) are shown to exist. In Theorem 4.3 it will be shown that the absence of constraints on δ_2 in (i) is required: For certain k, e.g. $k = \mathbb{Q}(i)$, all indecomposable division algebras D_2 are part of some nontrivial decomposition.

3. Residue field a number field

A prime field is a field that has no proper subfields, i.e., either a finite field of prime order, or the rational number field. All finite extensions of prime fields are absolutely stable, hence candidates for the residue field k in the above results. But if k is a finite field, then $\mathrm{Br}(F) \cong \mathbf{X}(k)$, hence F is absolutely stable, and all F-division algebras are indecomposable. In this section the remaining case, k a number field, is explored. The main result is Theorem 3.3 and proof, which gives a criterion for decomposability and a characterization of all decompositions of a given D. Several corollaries follow concerning the role of the semiramified subalgebras.

Other candidates for k are fields finitely generated of transcendence degree 1 over a finite field (the "global fields" that are not number fields). They are perfect with respect to primes not dividing their characteristics. For such primes, Index Formula 1.1 holds, and the theory is similar to that for number fields (except no Special Case!). They are not considered below, for simplicity's sake.

The following assumes a knowledge of the theory of central simple algebras and the Brauer group over a number field, as in [Re].

Let F be a rank one discretely Henselian field with residue field k a number field, and uniformizer t. For each prime $\mathfrak{q}$ of k, let $F_\mathfrak{q} \supset F$ denote the rank one discretely Henselian field with residue field $k_\mathfrak{q}$ and uniformizer t. Let the subscript $\mathfrak{q}$ denote "at the prime $\mathfrak{q}$". For example, let $\delta_\mathfrak{q}$ denote the image of δ in the homomorphism $\mathrm{Br}(F) \to \mathrm{Br}(F_\mathfrak{q})$. Obviously $\mathbf{e}(\delta_\mathfrak{q}) \,|\, \mathbf{e}(\delta)$ and $\mathbf{f}(\delta_\mathfrak{q}) \,|\, \mathbf{f}(\delta)$. Let $\mathrm{Loc}(\alpha)$ denote the ramification locus of an element $\alpha \in \mathrm{Br}(k)$, that is, the (finite) set of primes of k at which $\mathbf{i}(\alpha_\mathfrak{q})$ is nontrivial. If $\beta \in \mathrm{Br}(l)$ for a finite extension l/k, let $\mathrm{Loc}_k(\beta)$ denote the k-locus of primes divisible by the primes (of l) in $\mathrm{Loc}(\beta)$. As usual, the unramified character χ will sometimes be identified with its counterpart over k, and then the unramified extension $K\langle\chi\rangle/F$ is viewed as a finite extension of k.

Lemma 3.1: *Suppose $\delta \in \mathrm{Br}(F)$ has p-power index. Let R be the (possibly infinite) set of primes of k at which $\bar{\delta}$ has maximal index. Then*

$$p \,|\, \underline{\mathbf{c}}(\delta) \iff p \,|\, \mathbf{e}(\delta) \text{ and } \mathbf{e}(\delta_\mathfrak{q}) = 1 \;\; \forall \mathfrak{q} \in R.$$

Proof: Set $\delta = \alpha \dotplus (\chi, t)$. Abbreviate $\mathbf{e}(\delta)$ to $\mathbf{e}$, and similarly for the other invariants. By Theorem 1.3(iii), $p \,|\, \underline{\mathbf{c}} \iff p \,|\, \mathbf{e}$ and $\mathbf{i}((\alpha + (\chi, u))^{K\langle p\chi\rangle}) = \mathbf{f}$, $\forall\, u \in k^{\cdot}$. If $\bar{\delta}$ is nontrivial, then its ramification locus over $K\langle\chi\rangle$ is finite, and then R is finite. Conversely, if $\bar{\delta}$ is trivial, then R consists of all primes of k.

($\Rightarrow$): If $p \nmid \mathbf{e}$, then $p \nmid \underline{\mathbf{c}}$ by Theorem 1.3(iii).

Suppose $p \,|\, \mathbf{e}$ and $\mathbf{e}_\mathfrak{q} \neq 1$ for some $\mathfrak{q} \in R$. By Theorem 1.3(vi), $\underline{\mathbf{c}} \,|\, \mathbf{f}$ since k is a number field, so if $\mathbf{f} = 1$, then $\underline{\mathbf{c}} = 1$ and $p \nmid \underline{\mathbf{c}}$. Now assume $\mathbf{f} \neq 1$. Since $\mathfrak{q} \in R$, $\mathbf{f}_\mathfrak{q} = \mathbf{f} \neq 1$, so $\mathbf{i}(\alpha_\mathfrak{q}^{K\langle\chi_\mathfrak{q}\rangle}) \neq 1$. By local class field theory this implies $K\langle\chi_\mathfrak{q}\rangle \subset \Delta(\alpha_\mathfrak{q})$, hence $\alpha_\mathfrak{q}^{K\langle p\chi_\mathfrak{q}\rangle}$ has index $p \cdot \mathbf{f}$, hence $\mathbf{i}(\alpha^{K\langle p\chi\rangle}) = p \cdot \mathbf{f}$ by the local-global index formula for k a number field ([Re] (32.19)). Therefore $p \nmid \underline{\mathbf{c}}$ by Theorem 1.3(iii).

($\Leftarrow$): Suppose $p \,|\, \mathbf{e}$, and $\mathbf{e}_\mathfrak{q} = 1$ for all $\mathfrak{q} \in R$. By definition of R, $\mathbf{i}((\alpha + (\chi, u))_\mathfrak{q}^{K\langle\chi_\mathfrak{q}\rangle}) = \mathbf{f}$ for all $u \in k^{\cdot}$, and since $\chi_\mathfrak{q}$ is trivial, $\mathbf{i}((\alpha + (\chi, u))_\mathfrak{q}^{K\langle p\chi_\mathfrak{q}\rangle}) = \mathbf{f}$

for all $\mathfrak{q} \in R$. If $\mathfrak{p}$ is a prime not in R, then $\mathbf{i}((\alpha + (\chi, u))_{\mathfrak{p}}^{K\langle \chi_{\mathfrak{p}} \rangle}) \leq \mathbf{f}/p$ ($p \mid \mathbf{f}$ if R does not contain all primes of k), hence $\mathbf{i}((\alpha + (\chi, u))_{\mathfrak{p}}^{K\langle p\chi_{\mathfrak{p}} \rangle}) \leq \mathbf{f}$. Therefore $\mathbf{i}((\alpha + (\chi, u))_{\mathfrak{q}}^{K\langle p\chi_{\mathfrak{q}} \rangle}) \leq \mathbf{f}$ for all primes $\mathfrak{q}$ of k and for all $u \in k^{\cdot}$, hence by the local-global index formula again, $\mathbf{i}(\alpha + (\chi, u)^{K\langle p\chi \rangle}) = \mathbf{f}$ for all $u \in k^{\cdot}$. Therefore $p \mid \underline{\mathbf{c}}$ by Theorem 1.3(iii). ∎

COROLLARY 3.2: *There exist indecomposable F-division algebras of unequal period and index. The smallest examples have $(\mathbf{o}, \mathbf{i}) = (p^2, p^3)$, for any number field k.*

Proof: Let $\alpha \in \mathrm{Br}(k)$ have ramification locus $\{\mathfrak{q}_1, \mathfrak{q}_2\}$, with index p^2 (and opposing invariants) at each prime. Such an α exists by a theorem of Hasse ([Re] (32.13)). By the Grunwald–Wang Theorem, there exists a character χ of order p^2 that has order p at $\mathfrak{q}_1$ and $\mathfrak{q}_2$. Let $\delta = \alpha \dot{+} (\chi, t)$. Using (1.0.2), Index Formula 1.1, and the local-global index formula, compute $\mathbf{o}(\delta) = p^2$, and $\mathbf{i}(\delta) = p^2 \cdot p = p^3$. By Lemma 3.1, $p \nmid \underline{\mathbf{c}}(\delta)$, and so $\Delta(\delta)$ is indecomposable by Lemma 2.1.

By (2.1.3), $\mathbf{i}(\delta) \mid p^2 \cdot \mathbf{i}(p\delta)$, and if $\mathbf{o}(\delta) = p$, $\mathbf{i}(p\delta) = 1$. Therefore if $\mathbf{o}(\delta) = p$, then $\mathbf{o}(\delta) \neq \mathbf{i}(\delta)$ if and only if $\mathbf{i}(\delta) = p^2$. If $\mathbf{o}(\delta) = p$, then $\mathbf{i}(\alpha)$ and $\mathbf{i}((\chi, t))$ both divide p by (1.0.2), hence if $\mathbf{i}(\delta) = p^2$, then $\mathbf{i}(\delta) = \mathbf{i}(\alpha)\,\mathbf{i}((\chi, t))$, and $\mathbf{i}(\alpha) = \mathbf{i}((\chi, t)) = p$. This shows that $\Delta(\delta) \cong \Delta(\alpha) \otimes \Delta(\chi, t)$ is a (split) decomposition. Conclude that if $\mathbf{o}(\delta) = p$ and $\mathbf{o}(\delta) \neq \mathbf{i}(\delta)$, then $\Delta(\delta)$ is decomposable. As a result, the (p^2, p^3) example above is the smallest possible. ∎

If k is merely absolutely stable, and δ is stepped, then by Definition 1.2, $\bar{\mathbf{c}}(\delta) \nmid \mathbf{f}(\delta)$, but the values of $\underline{\mathbf{c}}(\delta)$ and $\bar{\mathbf{c}}(\delta)$ remain mysterious. However, if k is a number field, then by Theorem 1.3(vi), $\underline{\mathbf{c}}(\delta) = \mathbf{f}(\delta)$ and $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$. Thus for example $(\mathbf{f}(\delta), \underline{\mathbf{c}}(\delta))$ in Lemma 2.1 is always equal to $\underline{\mathbf{c}}(\delta)$. Theorem 2.2 gave necessary conditions on the decompositions of a given $\Delta(\delta)$. It will now be shown (as mentioned in Remark 2.2.3) by specializing to the case where k is a number field that Theorem 2.2 is best possible.

THEOREM 3.3: *Suppose $\delta \in \mathrm{Br}(F)$ has p-power index. Then $\Delta(\delta)$ is decomposable if and only if $p \mid \underline{\mathbf{c}}(\delta)$. If $\Delta(\delta)$ is decomposable, it is possible to characterize all existing decompositions in terms of local data.*

Suppose $p \mid \underline{\mathbf{c}}(\delta)$, and let n be a number such that $n \mid \underline{\mathbf{c}}(\delta)$.

(i) *If $p \mid n$, then there exists a nontrivial decomposition $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta_2)$ with δ_1 semiramified, $\mathbf{i}(\delta_1) = n$, and $\mathbf{e}(\delta_1) \mid \mathbf{e}(\delta_2)$ $(= \mathbf{e}(\delta))$.*

(ii) *If* $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$, *then there exists a nontrivial decomposition* $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta_2)$ *with* $\mathbf{e}(\delta_1) = n$, $\mathbf{e}(\delta_1) \,|\, \mathbf{e}(\delta_2)$ $(= \mathbf{e}(\delta))$, *and* δ_2 *semiramified.*

All decompositions of $\Delta(\delta)$ *fit the descriptions (i) and/or (ii). Thus either the decomposition has a semiramified factor of index dividing* $\underline{\mathbf{c}}(\delta)$, *or* $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$ *and there is a semiramified factor of index* $\mathbf{e}(\delta)$ *(or both).*

Proof: Note if $p \nmid \underline{\mathbf{c}}(\delta)$, then $\Delta(\delta)$ is indecomposable by Lemma 2.1.

The proof of the theorem starts with the following lemma on properties of a nontrivial decomposition of $\Delta(\delta)$ that always exists when $p \,|\, \underline{\mathbf{c}}(\delta)$.

LEMMA 3.4: *Assume the set-up of Theorem 3.3, with* $p \,|\, \underline{\mathbf{c}}(\delta)$ *and* n *a number such that* $p \,|\, n \,|\, \underline{\mathbf{c}}(\delta)$. *Set* $\delta = \alpha \dotplus (\chi, t)$. *Let* R *be the set of primes of* k *over which* $\bar{\delta}$ *has maximal index (as in Lemma 3.1). Then*

(i) *There exists a character* χ_1 *of order* n *satisfying*

$$(3.4.1) \qquad |(\chi - \chi_1)_{\mathfrak{q}}| \ \geq\ \frac{n}{\mathbf{f}(\delta)}\ \mathbf{i}((\alpha + (\chi, u))_{\mathfrak{q}}) \quad \forall \mathfrak{q} \in \mathrm{Loc}(\alpha + (\chi, u)),$$

for some $u \in k^{\cdot}$, *with equality when* $\mathfrak{q} \in R$.

(ii) *For any* χ_1 *and* u *as in* (3.4.1), *the semiramified algebra* $\Delta(\chi_1, t/u)$ *appears in a decomposition of* δ. *That is,*

$$D(\delta) \ \cong\ \Delta(\chi_1, t/u) \otimes \Delta(\delta - (\chi_1, t/u)).$$

(iii) *If* $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta_2)$ *is a decomposition of* $\Delta(\delta)$ *with* δ_1 *SR of index* $\mathbf{e}(\delta_1)$ *dividing* $\mathbf{e}(\delta_2)$, *then* $\mathbf{e}(\delta_1) \,|\, \underline{\mathbf{c}}(\delta)$, *and* $\delta_1 = (\chi_1, t/u)$ *for some unramified* χ_1 *and* $u \in k^{\cdot}$ *satisfying* (3.4.1).

Proof: By hypothesis, $p \,|\, \underline{\mathbf{c}}(\delta)$, and since k is a number field, $\underline{\mathbf{c}}(\delta) \,|\, \mathbf{f}(\delta)$ by Theorem 1.3(vi). As mentioned in the first paragraph of Lemma 3.1, $p \,|\, \mathbf{f}(\delta)$ implies R is finite. Therefore R is finite here.

Proof of (ii): Suppose χ_1 satisfies (3.4.1). Obviously $\delta = (\chi_1, t/u) + (\delta - (\chi_1, t/u))$. To prove this is a nontrivial multiplicative decomposition it suffices to show that each factor is nontrivial, and that

$$(3.4.2) \qquad \mathbf{i}(\delta) \ =\ \mathbf{i}((\chi_1, t/u)) \cdot \mathbf{i}(\delta - (\chi_1, t/u)),$$

i.e., that the indexes are multiplicative.

It is immediate that $(\chi_1, t/u)$ is nontrivial, since $p \,|\, n = \mathbf{i}((\chi_1, t/u))$ by hypothesis. By Theorem 1.3(i), $\underline{\mathbf{c}}(\delta) \,|\, \mathbf{e}(\delta)$, so the assumption $n \,|\, \underline{\mathbf{c}}(\delta)$ implies $n \,|\, \mathbf{e}(\delta)$. If $n < \mathbf{e}(\delta)$, then since $n = |\chi_1|$, $|\chi - \chi_1| = |\chi| = \mathbf{e}(\delta)$, by abelian group theory.

CLAIM: If $n = \mathbf{e}(\delta)$, then also $|\chi - \chi_1| = \mathbf{e}(\delta)$: For since $p \,|\, \underline{\mathbf{c}}(\delta)$ (by hypothesis), $\mathbf{e}(\delta_\mathfrak{q}) \equiv |\chi_\mathfrak{q}| = 1$ for all $\mathfrak{q}$ in R by Lemma 3.1. Thus $\mathbf{i}((\alpha + (\chi, u))_\mathfrak{q}^{K\langle\chi_\mathfrak{q}\rangle}) = \mathbf{i}((\alpha + (\chi, u))_\mathfrak{q})$ $\forall \mathfrak{q} \in R$. This expression is the formula for residue index of δ at $\mathfrak{q}$ ([Re] (31.9)), which is $\mathbf{f}(\delta)$, since $\mathfrak{q} \in R$. Therefore $\mathbf{i}((\alpha + (\chi, u))_\mathfrak{q}) = \mathbf{f}(\delta)$, so (3.4.1) becomes $|(\chi - \chi_1)_\mathfrak{q}| = n$ whenever $\mathfrak{q}$ is in R. But $|(\chi - \chi_1)_\mathfrak{q}| \,\big|\, |\chi - \chi_1|$, so this means $n \,\big|\, |\chi - \chi_1|$, and by the claim's hypothesis, $n = \mathbf{e}(\delta)$. Therefore $\mathbf{e}(\delta) \,\big|\, |\chi - \chi_1|$. Since $|\chi|$ and $|\chi_1|$ both divide $\mathbf{e}(\delta)$, $|\chi - \chi_1| \,\big|\, \mathbf{e}(\delta)$. Thus $|\chi - \chi_1| = \mathbf{e}(\delta)$, as claimed.

Since $\delta = \alpha \dotplus (\chi, t)$, an expansion shows that $\chi - \chi_1$ is the character of the alleged decomposition factor $\delta - (\chi_1, t/u)$ of δ. Thus $\mathbf{e}(\delta - (\chi_1, t/u)) = |\chi - \chi_1| = \mathbf{e}(\delta)$, by the above. Since $p \,|\, \underline{\mathbf{c}}(\delta)$ and $\underline{\mathbf{c}}(\delta) \,|\, \mathbf{e}(\delta)$ (by Theorem 1.3(i)), $\mathbf{e}(\delta)$ is nontrivial, hence by Index Formula 1.1, $\delta - (\chi_1, t/u)$ is nontrivial. It remains to demonstrate (3.4.2). Since $\mathbf{i}((\chi_1, t/u)) = n$ and $\mathbf{e}(\delta - (\chi_1, t/u)) = \mathbf{e}(\delta)$, (3.4.2) reduces to

$$(3.4.2)' \qquad \mathbf{f}(\delta) \;=\; n \cdot \mathbf{f}(\delta - (\chi_1, t/u)).$$

By the local-global index formula, $\mathbf{f}(\delta - (\chi_1, t/u)) = \sup_\mathfrak{q}\{\mathbf{f}((\delta - (\chi_1, t/u))_\mathfrak{q})\}$, where $\mathfrak{q}$ ranges over all primes. By definition, $\mathbf{f}((\delta - (\chi_1, t/u))_\mathfrak{q})$ is $\mathbf{i}((\alpha + (\chi, u))_\mathfrak{q}^{K\langle(\chi - \chi_1)_\mathfrak{q}\rangle})$, which by the index formula for a local field ([Re] (31.9)) is $\max\{1, \; \mathbf{i}((\alpha + (\chi, u))_\mathfrak{q})/|(\chi - \chi_1)_\mathfrak{q}| \;\}$. By (3.4.1), this number is less than or equal to $\max\{1, \mathbf{f}(\delta)/n\}$ for all $\mathfrak{q} \in \mathrm{Loc}(\alpha + (\chi, u))$, with equality if $\mathfrak{q} \in R$. By hypothesis $n \,|\, \underline{\mathbf{c}}(\delta)$, and so $n \,|\, \mathbf{f}(\delta)$ by Theorem 1.3(vi). Therefore $\max\{1, \mathbf{f}(\delta)/n\} = \mathbf{f}(\delta)/n$, hence $\mathbf{f}((\delta - (\chi_1, t/u))_\mathfrak{q}) \leq \mathbf{f}(\delta)/n$, with equality for $\mathfrak{q} \in R$. Therefore by the local-global index formula, $\mathbf{f}(\delta - (\chi_1, t/u)) = \mathbf{f}(\delta)/n$. This establishes (3.4.2)$'$, completing the proof of (ii).

Proof of (i): In [Br2] (Theorem 5.12) it is shown that

$$|\chi_\mathfrak{q}| \geq \frac{\underline{\mathbf{c}}(\delta)}{\mathbf{f}(\delta)} \cdot \mathbf{i}((\alpha + (\chi, u))_\mathfrak{q}) \quad \forall \mathfrak{q} \in \mathrm{Loc}(\mathbf{f}(\delta) \cdot (\alpha + (\chi, u)))$$

(i.e., for all $\mathfrak{q}$ at which the index of $\alpha + (\chi, u)$ exceeds $\mathbf{f}(\delta)$). Since $n \,|\, \underline{\mathbf{c}}(\delta)$ (by hypothesis), $|\chi_\mathfrak{q}| \geq \frac{n}{\mathbf{f}(\delta)} \cdot \mathbf{i}((\alpha + (\chi, u))_\mathfrak{q})$ $\forall \mathfrak{q} \in \mathrm{Loc}(\mathbf{f}(\delta) \cdot (\alpha + (\chi, u)))$. Therefore the first part of (3.4.1) is satisfied whenever $\mathfrak{q} \in \mathrm{Loc}(\mathbf{f}(\delta) \cdot (\alpha + (\chi, u)))$ and $|(\chi - \chi_1)_\mathfrak{q}| \geq |\chi_\mathfrak{q}|$. The difference set $\mathrm{Loc}(\frac{\mathbf{f}(\delta)}{p} \cdot (\alpha + (\chi, u))) \backslash \mathrm{Loc}(\mathbf{f}(\delta) \cdot (\alpha + (\chi, u)))$ consists of the primes at which the index of $\alpha + (\chi, u)$ is equal to $\mathbf{f}(\delta)$. At these primes, the first part of (3.4.1) becomes $|(\chi - \chi_1)_\mathfrak{q}| \geq n$. The remaining primes in $\mathrm{Loc}(\alpha + (\chi, u))$ belong to $\mathrm{Loc}(\alpha + (\chi, u)) \backslash \mathrm{Loc}(\frac{\mathbf{f}(\delta)}{p} \cdot (\alpha + (\chi, u)))$. At these $\mathfrak{q}$, $\mathbf{i}((\alpha + (\chi, u))_\mathfrak{q}) < \mathbf{f}(\delta)$, hence (3.4.1) is satisfied if $|(\chi - \chi_1)_\mathfrak{q}| \geq n$.

The second part of (3.4.1) is satisfied if whenever $\mathfrak{q} \in R$, $|(\chi - \chi_1)_\mathfrak{q}| = \frac{n}{\mathbf{f}(\delta)} \cdot \mathbf{i}((\alpha + (\chi, u))_\mathfrak{q})$. By definition of R, $\mathbf{f}(\delta_\mathfrak{q}) = \mathbf{f}(\delta)$ at such $\mathfrak{q}$. Therefore since $\mathbf{f}(\delta) \neq 1$ (shown above), $\frac{1}{\mathbf{f}(\delta)} \mathbf{i}((\alpha + (\chi, u))_\mathfrak{q}) = |\chi_\mathfrak{q}|$. By hypothesis, $p \,|\, \underline{\mathbf{c}}(\delta)$, so by Lemma 3.1, $|\chi_\mathfrak{q}| = 1$. Therefore for $\mathfrak{q} \in R$, (3.4.1) is satisfied if and only if $|(\chi_1)_\mathfrak{q}| = n$.

Thus the proof of (i) boils down to finding a character χ_1 of order n such that

(a) $|(\chi - \chi_1)_\mathfrak{q}| \geq |\chi_\mathfrak{q}|$ when $\mathfrak{q} \in \mathrm{Loc}(\mathbf{f}(\delta) \cdot (\alpha + (\chi, u)))$,

(b) $|(\chi - \chi_1)_\mathfrak{q}| \geq n$ when $\mathfrak{q} \in \mathrm{Loc}(\alpha + (\chi, u)) \backslash \mathrm{Loc}(\mathbf{f}(\delta) \cdot (\alpha + (\chi, u)))$, and

(c) $|(\chi_1)_\mathfrak{q}| = n$ when $\mathfrak{q} \in R$.

Clearly (a) is satisfied if $|(\chi_1)_\mathfrak{q}| = 1$. If $|\chi_\mathfrak{q}| < n$ (e.g. if $\mathfrak{q} \in R$), then (b) and (c) are satisfied if $|(\chi_1)_\mathfrak{q}| = n$, by abelian group theory. If $|\chi_\mathfrak{q}| > n$, then (b) is satisfied if $|(\chi_1)_\mathfrak{q}| = 1$. A character χ_1 of order n with these local orders at the (finite) set of primes in $\mathrm{Loc}(\alpha + (\chi, u))$ exists by the weak Grunwald–Wang Theorem ([AT] p. 105). This completes the proof of (i).

Proof of (iii): Suppose $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta_2)$ is a nontrivial decomposition of $\Delta(\delta)$, with δ_1 SR of index $\mathbf{e}(\delta_1)$, and $\mathbf{e}(\delta_1) \,|\, \mathbf{e}(\delta_2)$. Since δ_1 is SR, it has the form $(\chi_1, t/u)$ for some χ_1 and $u \in k^{\cdot}$. By (2.2.1), $\mathbf{e}(\delta_1) \,|\, \mathbf{e}(\delta_2)$ implies $\mathbf{e}(\delta_1) \,|\, \underline{\mathbf{c}}(\delta)$. Therefore to prove (iii) it suffices to show χ_1 satisfies (3.4.1) with n replaced by $\mathbf{e}(\delta_1)$.

By (2.1.4), $\mathbf{f}(\delta_2) = \mathbf{f}(\delta)/\mathbf{e}(\delta_1)$. Since $\delta_2 = \delta - \delta_1 = \alpha + (\chi, u) \dot{+} (\chi - \chi_1, t/u)$, the formula for $\mathbf{f}(\delta_2)$ is $\mathbf{f}(\delta_2) = \sup_\mathfrak{q}\{1, \mathbf{i}((\alpha + (\chi, u))_\mathfrak{q})/|(\chi - \chi_1)_\mathfrak{q}|\}$, where $\mathfrak{q}$ ranges over all primes in the locus of $\alpha + (\chi, u)$. Setting these 2 expressions for $\mathbf{f}(\delta_2)$ equal yields the inequality $|(\chi - \chi_1)_\mathfrak{q}| \geq \frac{\mathbf{e}(\delta_1)}{\mathbf{f}(\delta)} \cdot \mathbf{i}((\alpha + (\chi, u))_\mathfrak{q})$ $\forall \mathfrak{q} \in \mathrm{Loc}(\alpha + (\chi, u))$. If there is equality $\forall \mathfrak{q} \in R$, then (iii) is proved. But if $\mathfrak{q} \in R$, then (as noted previously) the hypothesis $p \,|\, \underline{\mathbf{c}}(\delta)$ implies $|\chi_\mathfrak{q}| = 1$ by Lemma 3.1, hence $|(\chi - \chi_1)_\mathfrak{q}| = |(\chi_1)_\mathfrak{q}|$, and $\mathbf{f}(\delta) = \mathbf{i}((\alpha + (\chi, u))_\mathfrak{q}^{K(\chi_\mathfrak{q})}) = \mathbf{i}((\alpha + (\chi, u))_\mathfrak{q})$. Therefore the inequality becomes $|(\chi_1)_\mathfrak{q}| \geq \mathbf{e}(\delta_1) \equiv |\chi_1|$, and equality is forced. This completes the proof of Lemma 3.4. ∎

By Lemma 3.4(i) and (ii), $p \mid \underline{\mathbf{c}}(\delta)$ implies $\Delta(\delta)$ is decomposable. Since $\Delta(\delta)$ decomposable implies $p \mid \underline{\mathbf{c}}(\delta)$ by Lemma 2.1, this proves the first statement of the theorem.

By Lemma 3.4(iii) the construction of Lemma 3.4(ii) exhausts the list of decompositions of the form $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta_2)$ with $\mathbf{e}(\delta_1) \mid \mathbf{e}(\delta_2)$ and δ_1 SR. It remains to consider those decompositions (if they exist) where δ_1 is **not** SR. By Theorem 2.2, δ_2 is then SR.

LEMMA 3.5: *Assume the set-up of Theorem 3.3, with $p \mid \underline{\mathbf{c}}(\delta)$ and n a number such that $n \mid \underline{\mathbf{c}}(\delta)$, but $n \neq \underline{\mathbf{c}}(\delta)$. Set $\delta = \alpha \dotplus (\chi, t)$. Let $u \in k^\cdot$ be such that $c_{1/u}(\delta, t) = \bar{\mathbf{c}}(\delta)$. Then*

(i) *There exists a character χ_1 of order n satisfying*

$$(3.5.1) \qquad |(\chi_1)_\mathfrak{q}| \geq \frac{n}{\mathbf{f}(\delta)} \, \mathbf{i}((\alpha + (\chi, u))_\mathfrak{q}) \qquad \forall \mathfrak{q} \in \mathrm{Loc}(\alpha + (\chi, u))$$

if and only if $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$.

(ii) *For any χ_1 satisfying (3.5.1), $\Delta(\delta_1) := \Delta(\delta - (\chi - \chi_1, t/u))$ is not SR, and it appears in a decomposition of $\Delta(\delta)$. That is,*

$$\Delta(\delta) \cong \Delta(\delta - (\chi - \chi_1, t/u)) \otimes \Delta(\chi - \chi_1, t/u).$$

(iii) *If $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta_2)$ is a nontrivial decomposition of $\Delta(\delta)$ such that δ_1 is not SR and $\mathbf{e}(\delta_1) \mid \mathbf{e}(\delta_2)$, then $\delta_1 = \delta - (\chi - \chi_1, t/u)$ for some χ_1 satisfying (3.5.1), and some $u \in k^\cdot$ such that $c_{1/u}(\delta, t) = \bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$. Furthermore, $\mathbf{e}(\delta_1) \mid \underline{\mathbf{c}}(\delta)$, $\mathbf{e}(\delta_1) \neq \underline{\mathbf{c}}(\delta)$, and the factor δ_2 is SR of index $\mathbf{e}(\delta)$.*

Proof: Since k is a number field, $\underline{\mathbf{c}}(\delta) \mid \mathbf{f}(\delta)$, by Theorem 1.3(vi). Therefore the hypothesis $p \mid \underline{\mathbf{c}}(\delta)$ implies $\mathbf{f}(\delta) \neq 1$, and the hypotheses $n \mid \underline{\mathbf{c}}(\delta), n \neq \underline{\mathbf{c}}(\delta)$ imply $n \mid \mathbf{f}(\delta), n \neq \mathbf{f}(\delta)$.

By Definition 1.2, $\bar{\mathbf{c}}(\delta)$ is the least common multiple of the various $c_u(\delta, t)$, so it is always possible to find a $u \in k^\cdot$ such that $c_{1/u}(\delta, t) = \bar{\mathbf{c}}(\delta)$.

Proof of (ii): Suppose χ_1 is a character of order n such that $n \mid \underline{\mathbf{c}}(\delta)$, $n \neq \underline{\mathbf{c}}(\delta)$, and (3.5.1) holds. Set $\delta_1 = \delta - (\chi - \chi_1, t/u)$. Expanding δ yields $\delta_1 = \alpha + (\chi, u) \dotplus (\chi_1, t/u)$, and $\delta - \delta_1 = (\chi - \chi_1, t/u)$.

CLAIM: The condition (3.5.1) implies that $K\langle\chi_1\rangle \subset \Delta(\alpha + (\chi, u))$, $\mathbf{i}(\alpha + (\chi, u)) = \mathbf{f}(\delta)$, and $c_{1/u}(\delta, t) = \bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$. For by the local-global index formula and the formula for computing index over a local field,

$$\mathbf{i}(\alpha + (\chi, u)^{K\langle\chi_1\rangle}) = \sup_\mathfrak{q}\{1, \ \mathbf{i}((\alpha + (\chi, u))_\mathfrak{q})/|(\chi_1)_\mathfrak{q}|\}.$$

Applying (3.5.1) to this sup shows the sup is less than or equal to $\sup_{\mathfrak{q}}\{1, \mathbf{f}(\delta)/n\}$, and the latter equals $\mathbf{f}(\delta)/n$ since $n \mid \mathbf{f}(\delta)$. Thus $\mathbf{i}((\alpha+(\chi, u))^{K\langle\chi_1\rangle}) \leq \mathbf{f}(\delta)/n \equiv \mathbf{f}(\delta)/|\chi_1|$. On the other hand,

$$\mathbf{f}(\delta)/|\chi_1| \equiv \mathbf{i}((\alpha+(\chi, u))^{K\langle\chi\rangle})/|\chi_1| \leq \mathbf{i}(\alpha+(\chi, u))/|\chi_1| \leq \mathbf{i}((\alpha+(\chi, u))^{K\langle\chi_1\rangle}),$$

by the basic theory. Therefore $\mathbf{i}((\alpha+(\chi, u))^{K\langle\chi_1\rangle}) = \mathbf{i}(\alpha+(\chi, u))/|\chi_1| = \mathbf{i}((\alpha+(\chi, u))^{K\langle\chi\rangle})/|\chi_1|$. The first of these two equalities shows $K\langle\chi_1\rangle \subset \Delta(\alpha+(\chi, u))$, and the second shows $\mathbf{i}(\alpha+(\chi, u)) = \mathbf{f}(\delta)$. This proves the second part of the claim, and shows that $c_{1/u}(\delta, t) = \mathbf{e}(\delta)$ by Theorem 1.3(ii). Since $c_{1/u}(\delta, t) = \bar{\mathbf{c}}(\delta)$ by hypothesis, $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$. This completes the proof of the claim.

As always, the expression $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta - \delta_1)$ is a nontrivial decomposition of $\Delta(\delta)$ if and only if the factors on the right are nontrivial, and their indexes are multiplicative, i.e., if and only if

$$\mathbf{i}(\delta) = \mathbf{i}(\delta_1) \cdot \mathbf{i}((\chi - \chi_1, t/u)), \tag{3.5.2}$$

and $\mathbf{i}(\delta_1), \mathbf{i}((\chi-\chi_1, t/u)) \neq 1$. By Theorem 1.3(i), $\underline{\mathbf{c}}(\delta) \mid |\chi|$. Therefore, since $n < \underline{\mathbf{c}}(\delta)$, necessarily $|\chi_1| < |\chi|$ (possibly $|\chi_1| = 1$). Thus $|\chi - \chi_1| = |\chi| = \mathbf{e}(\delta)$. Since $\mathbf{i}((\chi - \chi_1, t/u)) = |\chi - \chi_1|$, $(\chi - \chi_1, t/u)$ is nontrivial, and (3.5.2) becomes

$$\mathbf{f}(\delta) = \mathbf{i}(\delta_1), \tag{3.5.2$'$}$$

by Index Formula 1.1. Since $\mathbf{f}(\delta) \neq 1$, establishing (3.5.2)′ will also establish the nontriviality of δ_1. By the claim, $\mathbf{f}(\delta) = \mathbf{i}(\alpha+(\chi, u))$. By Index Formula 1.1, $\mathbf{i}(\delta_1) = |\chi_1|\,\mathbf{i}((\alpha+(\chi, u))^{K\langle\chi_1\rangle})$, and since $K\langle\chi_1\rangle \subset \Delta(\alpha+(\chi, u))$ by the claim, $\mathbf{i}(\delta_1) = \mathbf{i}(\alpha+(\chi, u))$. Thus $\mathbf{f}(\delta) = \mathbf{i}(\delta_1) \neq 1$. Therefore $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta-\delta_1)$ is a nontrivial decomposition.

It remains to show δ_1 is not SR. This now follows from the hypothesis $n = |\chi_1| < \underline{\mathbf{c}}(\delta)$: For by Theorem 1.3(vi), $\underline{\mathbf{c}}(\delta) \mid \mathbf{f}(\delta)$ (since k is a number field), and it has just been demonstrated that $\mathbf{f}(\delta) = \mathbf{i}(\delta_1)$. Therefore $|\chi_1| \equiv \mathbf{e}(\delta_1) < \mathbf{i}(\delta_1)$, hence $\mathbf{f}(\delta_1) \neq 1$ by Index Formula 1.1. This completes the proof of (ii).

Proof of (i): By the claim in the proof of (ii), the existence of a χ_1 satisfying (3.5.1) implies $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$.

Conversely, suppose $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$. Condition (3.5.1) is satisfied by a character χ_1 of order n with local degrees $|(\chi_1)_{\mathfrak{q}}| = \frac{n}{\mathbf{f}(\delta)} \cdot \mathbf{i}((\alpha+(\chi, u))_{\mathfrak{q}})\ \forall \mathfrak{q} \in \mathrm{Loc}(\alpha+(\chi, u))$.

The weak Grunwald–Wang Theorem shows there exists a character of any order with a finite set of prescribed local orders, as long as the local orders divide the (global) order. Thus this χ_1 exists if $\frac{n}{\mathbf{f}(\delta)} \cdot \mathbf{i}((\alpha + (\chi, u))_{\mathfrak{q}}) \leq n$ $\forall\mathfrak{q}$, i.e., $\mathbf{i}((\alpha + (\chi, u))_{\mathfrak{q}})/\mathbf{f}(\delta) \leq 1$.

By hypothesis, $c_{1/u}(\delta, t) = \bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$. Therefore by Theorem 1.3(ii), $\mathbf{i}(\alpha + (\chi, u)) = \mathbf{f}(\delta)$. Consequently, by the local-global index formula,

$$\mathbf{i}((\alpha + (\chi, u))_{\mathfrak{q}})/\mathbf{f}(\delta) \leq 1 \quad \text{for all } \mathfrak{q} \in \mathrm{Loc}(\alpha + (\chi, u)),$$

as desired. This completes the proof of (i).

Proof of (iii): Suppose $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta_2)$ is a decomposition of $\Delta(\delta)$, with δ_1 not SR, and $\mathbf{e}(\delta_1) \,|\, \mathbf{e}(\delta_2)$. Since δ_1 is not SR, δ_2 is SR by Theorem 2.2. By (2.2.1), $\mathbf{e}(\delta_2) = \mathbf{e}(\delta)$, hence δ_2 is SR of index $\mathbf{e}(\delta)$, as claimed.

By Theorem 2.2, $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$, as claimed, and $\mathbf{i}(\delta_1) = \mathbf{o}(\delta_1) = \mathbf{f}(\delta)$. Since δ_1 is not SR, its ramification index must be strictly less than its index, by Index Formula 1.1. Therefore $\mathbf{e}(\delta_1) < \mathbf{i}(\delta_1) = \mathbf{o}(\delta_1)$. By Lemma 2.1, $(\mathbf{o}(\delta_1), \mathbf{o}(\delta_2)) \,|\, \underline{\mathbf{c}}(\delta)$, hence $\mathbf{e}(\delta_1) < \underline{\mathbf{c}}(\delta)$, as claimed.

It remains to show that the δ_1 in this decomposition has the form of part (ii), and satisfies (3.5.1). Let χ_1 be the character of δ_1. Since δ_2 is SR, $\delta_2 = (\chi - \chi_1, t/u)$ for some $u \in k^{\cdot}$. Since $\delta = \alpha + (\chi, t)$, $\delta_1 = \delta - \delta_2 = \alpha - (\chi - \chi_1, 1/u) \dotplus (\chi, t) - (\chi - \chi_1, t) = \alpha + (\chi, u) \dotplus (\chi_1, t/u)$. By Lemma 1.4, the conditions $\mathbf{i}(\delta_1) = \mathbf{o}(\delta_1)$ and δ_1 not SR together imply that $K\langle\chi_1\rangle \subset \Delta(\alpha + (\chi, u))$. Therefore $\mathbf{f}(\delta_1) \equiv \mathbf{i}((\alpha + (\chi, u))^{K\langle\chi_1\rangle}) = \mathbf{i}(\alpha + (\chi, u))/\mathbf{e}(\delta_1)$ by the general theory, and by Index Formula 1.1, $\mathbf{i}(\delta_1) = \mathbf{i}(\alpha + (\chi, u))$. By the above, $\mathbf{i}(\delta_1) = \mathbf{f}(\delta)$, so $\mathbf{i}(\alpha + (\chi, u)) = \mathbf{f}(\delta)$.

Since $\mathbf{i}(\alpha + (\chi, u)) = \mathbf{f}(\delta)$, $c_{1/u}(\delta, t) = \bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$ by Theorem 1.3(i) and (ii). It remains to show that χ_1 satisfies (3.5.1). By the local-global index formula, $\mathbf{i}((\alpha + (\chi, u))_{\mathfrak{q}})/|(\chi_1)_{\mathfrak{q}}| \leq \mathbf{f}(\delta_1)$ $\forall\mathfrak{q} \in \mathrm{Loc}(\alpha + (\chi, u))$. By Index Formula 1.1, $\mathbf{f}(\delta_1) = \mathbf{i}(\delta_1)/\mathbf{e}(\delta_1)$, and this is $\mathbf{f}(\delta)/\mathbf{e}(\delta_1)$ by above. Thus $\mathbf{i}((\alpha + (\chi, u))_{\mathfrak{q}})/|(\chi_1)_{\mathfrak{q}}| \leq \mathbf{f}(\delta)/\mathbf{e}(\delta_1)$ $\forall\mathfrak{q} \in \mathrm{Loc}(\alpha + (\chi, u))$. Rewriting produces (3.5.1), and finishes the proof of the lemma. ∎

Continue proof of Theorem 3.3. By Lemma 3.4(iii) and Lemma 3.5(iii), it is possible to characterize all existing decompositions in terms of local data, namely the conditions (3.4.1) and (3.5.1): For by Lemma 2.1(i), every decomposition of $\Delta(\delta)$ has form $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta_2)$ with $\mathbf{e}(\delta_1) \,|\, \mathbf{e}(\delta_2)$. One of the factors is SR by Theorem 2.2. If δ_1 is SR, then by Lemma 3.4(iii), the decomposition is accounted for by the construction of Lemma 3.4(ii). If δ_1 is not SR then by Lemma 3.5(iii), the decomposition is accounted for by the construction of Lemma 3.5(ii). This exhausts all possibilities.

The proofs of (i) and (ii) of the theorem are immediate from Lemma 3.4(i) and (ii) and Lemma 3.5(i) and (ii), respectively.

Finally, it remains to show all decomposition of $\Delta(\delta)$ fit the descriptions of (i) or (ii) of the theorem. It has been proved already that the constructions of Lemmas 3.4 and 3.5 exhaust the possibilities, so it suffices to consider only those constructions. But if a decomposition is produced by Lemma 3.4, then the factor δ_1 of that lemma is SR of index dividing $\underline{\mathbf{c}}(\delta)$, so it fits the description (i) of the theorem. If a decomposition is produced by Lemma 3.5, then the factor δ_2 of that lemma is SR of index $\mathbf{e}(\delta)$. By 3.5(i), $\mathbf{e}(\delta) = \bar{\mathbf{c}}(\delta)$, and by hypothesis, $\mathbf{e}(\delta_1)$ divides $\underline{\mathbf{c}}(\delta)$. Therefore this decomposition fits the description (ii) of the theorem. This completes the proof. ∎

Remarks 3.5.3: (i) If $p \cdot \mathbf{f}(\delta) < \bar{\mathbf{c}}(\delta)$ (so δ is stepped), then there is a gap in the range of indexes assumed by the various semiramified factors. For by Theorem 1.3(vi), $p \cdot \mathbf{f}(\delta) < \bar{\mathbf{c}}(\delta)$ implies $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$ and $\underline{\mathbf{c}}(\delta) = \mathbf{f}(\delta)$. Hence $\Delta(\delta)$ has a semiramified factor of index $\mathbf{e}(\delta)$ by Theorem 3.3(ii), and by Theorem 3.3(i), $\Delta(\delta)$ has semiramified factors of each index dividing $\underline{\mathbf{c}}(\delta) = \mathbf{f}(\delta)$. By the last statement of the theorem, this exhausts the possible indexes of semiramified factors. Therefore there are none with index strictly between $\mathbf{f}(\delta)$ and $\mathbf{e}(\delta)$. In terms of division algebras, this shows that there can be a gap in the range of indexes of semiramified F-subalgebras of a given (decomposable) F-division algebra. This contrasts with the situation for F-field extensions, where there is never a gap in the range of degrees of totally ramified subextensions.

(ii) As mentioned above, this theorem acts as a compliment to the results of Section 2 by proving the existence of the decompositions allowed there.

The next corollary lists the decomposability criteria proved so far.

COROLLARY 3.6: *Suppose $\delta \in \mathrm{Br}(F)$ has p-power index. Let R be the (nonempty) set of primes of k at which $\mathbf{f}(\delta) = \mathbf{f}(\delta_{\mathfrak{q}})$. Then the following are equivalent:*

(i) *$\Delta(\delta)$ is decomposable.*

(ii) *$\Delta(\delta)$ properly contains a semiramified F-division algebra.*

(iii) $\Delta(\delta)$ *has a totally ramified subfield, but is not itself semiramified.*
(iv) $p \,|\, \underline{\mathbf{c}}(\delta)$.
(v) $\mathbf{i}(p\delta) \neq \frac{1}{p}\,\mathbf{i}(\delta)$.
(vi) $p \,|\, \mathbf{e}(\delta)$ *and* $\mathbf{e}(\delta_{\mathfrak{q}}) = 1$ $\forall \mathfrak{q} \in R$.

Proof: The equivalence (i)$\Leftrightarrow$(iv) is stated in the theorem. The implication (i)$\Rightarrow$(ii) follows by Theorem 2.2, which says one of the factors in any nontrivial decomposition of $\Delta(\delta)$ is SR. If $\Delta(\delta)$ has a proper SR F-division subalgebra, then obviously it is decomposable and has a TR F-subfield. Moreover, SR algebras are indecomposable, since they have equal period and index, so $\Delta(\delta)$ cannot itself be SR. Therefore (ii)$\Rightarrow$(iii). If $\Delta(\delta)$ has a TR subfield, then $p \,|\, \bar{\mathbf{c}}(\delta)$ by Definition 1.2. If in addition $\Delta(\delta)$ is not SR, then since $p \,|\, \bar{\mathbf{c}}(\delta)$, $p \,|\, \underline{\mathbf{c}}(\delta)$ by Theorem 1.3(v) and (vi). Therefore (iii)$\Rightarrow$(iv), and so (i)–(iv) are equivalent. If $p \,|\, \underline{\mathbf{c}}(\delta)$, then $p \,|\, \mathbf{f}(\delta)$ by Theorem 1.3(vi), hence by Claim (2.1.2), $\mathbf{i}(p\delta) = \frac{1}{p^2}\,\mathbf{i}(\delta)$. Therefore (iv)$\Rightarrow$(v). Conversely, by (2.1.3) multiplication by p drops the index of δ by either p or p^2. By Claim (2.1.2) again, $\mathbf{i}(p\delta) = \frac{1}{p^2}\,\mathbf{i}(\delta)$ implies $p \,|\, \underline{\mathbf{c}}(\delta)$. Therefore (v)$\Rightarrow$(iv). The equivalence (iv)$\Leftrightarrow$(vi) is Lemma 3.1. ∎

The next corollary explicates the connection between decomposability and unequal period and index over F.

Corollary 3.7: *Suppose D/F has p-power index. Then*

$$\mathbf{i}(D) \neq \mathbf{o}(D) \iff D^{\otimes n} \text{ is decomposable for some } n.$$

Proof: The forward implication follows from the observation that if $\mathbf{i}(D) > \mathbf{o}(D)$, multiplication by some m must reduce the index by more than m. Letting np be the minimal such m (necessarily $np \,|\, \mathbf{o}(D)$), find that $\mathbf{i}(D^{\otimes np}) = \frac{1}{p^2}\,\mathbf{i}(D^{\otimes n})$, hence $D^{\otimes n}$ is decomposable by Corollary 3.6.

For the reverse, note that if $D^{\otimes n}$ is decomposable, then $D^{\otimes n}$ is nontrivial, and $\mathbf{i}(D^{\otimes n}) > \mathbf{o}(D^{\otimes n})$. By (2.1.1), $\mathbf{i}(D) \geq n\,\mathbf{i}(D^{\otimes n})$, and since $\mathbf{o}(D) = n\mathbf{o}(D^{\otimes n})$, this implies $\mathbf{i}(D) > \mathbf{o}(D)$. ∎

Remark 3.7.1: This corollary gives a type of explanation for index behavior over F that deviates from period behavior. For suppose $D \cong D_1 \otimes D_2$ is a nontrivial decomposition. By dimension count, $\mathbf{i}(D) = \mathbf{i}(D_1)\,\mathbf{i}(D_2)$, while by abelian group theory, $\mathbf{o}(D) \,|\, lcm\{\,\mathbf{o}(D_1), \mathbf{o}(D_2)\,\}$. Since period divides index, $lcm\{\,\mathbf{o}(D_1), \mathbf{o}(D_2)\,\} \,|\, lcm\{\,\mathbf{i}(D_1), \mathbf{i}(D_2)\,\}$, and since D_1 and D_2 are nontrivial, $lcm\{\,\mathbf{i}(D_1), \mathbf{i}(D_2)\,\} < \mathbf{i}(D_1)\,\mathbf{i}(D_2)$. Therefore when D is decomposable, $\mathbf{o}(D) <$

$\mathbf{i}(D)$ is forced by the basic theory. If $D^{\otimes n}$ is decomposable, not D, then by (2.1.1), which is also part of the basic theory, again $\mathbf{o}(D) < \mathbf{i}(D)$ is forced. In this way decomposability provides an explanation for unequal index and period in terms of the basic structure theory of division algebras. By Corollary 3.7, no other explanation is needed over F.

When $\mathbf{X}(k)$ has elements of all orders (i.e., k has cyclic extensions of all degrees), as is the case when k is a number field, every totally ramified F-extension is maximal in some semiramified F-division algebra. For every totally ramified extension of degree n has form $F(\sqrt[n]{ut})$ for some $u \in k^\cdot$, and if $\chi \in \mathbf{X}(k)$ is a character of order n, then $F(\sqrt[n]{ut})$ is a maximal subfield of the division algebra $\Delta(\chi^F, ut)$. The next result shows that this link between totally ramified F-extensions and semiramified F-division algebras endures with respect to containment in division algebras. That is, existence of a totally ramified F-subfield of a given D is equivalent to existence of a semiramified F-subalgebra of D. On the other hand, Corollary 3.9 shows it endures in a weak form: Not every totally ramified F-subfield is contained in a semiramified F-subalgebra. In both corollaries it is convenient to use the language of division algebras rather than that of Brauer elements.

COROLLARY 3.8: *An F-division algebra D of p-power index contains a totally ramified F-extension if and only if it contains a semiramified F-division algebra.*

Proof: If D contains a nontrivial SR F-division algebra, then it contains the (nonempty) set of TR F-extensions of that algebra. Conversely, if D contains a TR F-extension, then it may be SR, in which case D (trivially) contains a SR F-division algebra. If D is not SR, then it then contains a SR division algebra by Corollary 3.6. ∎

COROLLARY 3.9: *Let D be an F-division algebra of p-power index.*

(i) *Every totally ramified subfield of D of degree n is contained in a semiramified subalgebra of D if and only if $n \mid \mathbf{f}(D)$, $n = \mathbf{e}(D)$, or D is semiramified.*

(ii) *Suppose E is a totally ramified subfield of D of degree n. Then E is maximal in a semiramified subalgebra if and only if $n \mid \mathbf{f}(D)$ or $n = \mathbf{e}(D)$.*

Proof: Let E be a TR subfield of D of degree n. Since n is already the degree of a TR subfield, $n \mid \underline{\mathbf{c}}(D)$ is equivalent to $n \mid \mathbf{f}(D)$: The forward implication is immediate, since by Theorem 1.3(vi), $\underline{\mathbf{c}}(D) \mid \mathbf{f}(D)$ when k is a number field. The reverse is clear if $\underline{\mathbf{c}}(D) = \mathbf{f}(D)$, and by Theorem 1.3(v), if $\underline{\mathbf{c}}(D) < \mathbf{f}(D)$

then $\underline{\mathbf{c}}(D) = \bar{\mathbf{c}}(D)$, hence all TR subfields have degree dividing $\underline{\mathbf{c}}(D)$ since by Definition 1.2 they all have degree dividing $\bar{\mathbf{c}}(D)$.

By Lemma 2.1(i), all SR F-subalgebras of D have index dividing $\underline{\mathbf{c}}(D)$ and/or index equal to $\mathbf{e}(D)$. Therefore by the basic theory, a maximal E must have degree dividing $\underline{\mathbf{c}}(D)$ or degree equal to $\mathbf{e}(D)$. This is equivalent to $n \mid \mathbf{f}(D)$ or $n = \mathbf{e}(D)$, by the remarks of the first paragraph. This proves the forward implication of (ii).

By Lemma 3.4, D has a SR factor $\Delta(\chi, t/u)$ of any index m dividing $\underline{\mathbf{c}}(D)$, for any $u \in k^{\cdot}$. Since $\Delta(\chi_1, t/u)$ has maximal subfield $F(\sqrt[m]{t/u})$ and all TR extensions are of form $F(\sqrt[m]{t/u})$ for some $u \in k^{\cdot}$, this proves the reverse implications of both (i) and (ii) in the case $n \mid \underline{\mathbf{c}}(D)$, hence in the case $n \mid \mathbf{f}(D)$ by the remarks of the first paragraph.

The reverse implication of (i) for D SR is trivial.

To prove the rest of the reverse implications of (i) and (ii), suppose $n = \mathbf{e}(D)$. Then since D has a TR subfield (E) of degree $\mathbf{e}(D)$, $\bar{\mathbf{c}}(D) = \mathbf{e}(D)$ by Definition 1.2, hence by Lemma 1.4 and proof, D has a decomposition $D \cong A \otimes \Delta(\chi, t)$ for some (possibly trivial) unramified A and some choice of uniformizer t. Suppose $E = F(\sqrt[n]{ut})$, $u \in k^{\cdot}$. It is shown in [Br2] (Lemma 3.4) that since $[E : F] = \mathbf{e}(D)$, $E \subset D$ if and only if $u \in \mathrm{N}(\mathbf{f}(D) \cdot \chi)$. On the other hand, the similarity $D \sim (A \otimes \Delta(\chi, 1/u)) \otimes (\Delta(\chi, ut))$ is an isomorphism, hence a (split) decomposition if and only if $\mathbf{i}(A \otimes \Delta(\chi, 1/u)) = \mathbf{f}(D)$, since $\mathbf{i}((\chi, ut)) = \mathbf{e}(D)$. Since $D \cong A \otimes \Delta(\chi, t)$, already $\mathbf{i}(A) = \mathbf{f}(D)$, and always $\mathbf{f}(D) \mid \mathbf{i}(A \otimes \Delta(\chi, t/u))$. Therefore $\mathbf{i}(A \otimes \Delta(\chi, 1/u)) = \mathbf{f}(D)$ if and only if $\mathbf{i}(\Delta(\chi, 1/u)) \mid \mathbf{f}(D)$, i.e., $\Delta(\mathbf{f}(D) \cdot \chi, 1/u) \sim F$. By the basic theory, this is equivalent to $u \in \mathrm{N}(\mathbf{f}(D) \cdot \chi)$. The conclusion is $F(\sqrt[n]{ut}) \subset D$ implies $\Delta(\chi, ut) \subset D$, which proves the rest of the reverse implications of (i) and (ii).

It remains to prove the forward implication of (i). Suppose $n \nmid \mathbf{f}(D)$ and $n \neq \mathbf{e}(D)$, and E is TR of degree n.

CLAIM: E is contained in a TR subfield of degree $\mathbf{e}(D)$ if and only if E is contained in a SR subalgebra. If E is contained in a TR subfield L of degree $\mathbf{e}(D)$, then by the reverse implication of (i), E is contained in a SR subalgebra of D, namely the one containing L. Conversely, if E is not contained in a TR subfield of degree $\mathbf{e}(D)$, then E is not contained in any SR subalgebra of D: For if it was, then the degree of the SR subalgebra would necessarily be of index $\mathbf{e}(D)$, since there are no SR subalgebras of index strictly between $\mathbf{f}(D)$ and $\mathbf{e}(D)$

by Theorem 3.3, and already $n \nmid \mathbf{f}(D)$. Therefore since the degree of E is strictly less than $\mathbf{e}(D)$, E would not be maximal. But in [Br2] (Proposition 5.5), it is proved that all TR subfields of a SR division algebra are contained in TR maximal subfields. Therefore E would be contained in a TR subfield of degree $\mathbf{e}(D)$, a contradiction.

Thus the TR subfields of degree $n \nmid \mathbf{f}(D)$ not contained in SR subalgebras are exactly those not contained in TR subfields of degree $\mathbf{e}(D)$. There is a natural inclusion of the set of isomorphism classes of TR subfields of D of degree n that are subextensions of subfields of degree $\mathbf{e}(D)$ into the set of isomorphism classes of TR subfields of D of degree n. It is shown in [Br2] (Proposition 4.2) that the cokernel of this set map is an orbit of the group $\mathrm{N}(\mathbf{f}(D)\cdot\chi)k^{\cdot n/\mathbf{f}(D)}/\mathrm{N}(\mathbf{f}(D)\cdot\chi)k^{\cdot n}$ (note $\mathbf{f}(D) \mid n$ by hypothesis). By Prop. 5.14 of [Br2] this group is trivial if and only if D is SR (this uses k a number field). The conclusion is that if $n \nmid \mathbf{f}(D)$, $n \neq \mathbf{e}(D)$, and D is not SR, then there exist TR subfields of degree n that are not contained in any SR subalgebra. This completes the proof of (i). ∎

Remark 3.9.1: By Corollary 3.9, every TR subfield of D is contained in a SR subalgebra if and only if D is stubbed ($\bar{\mathbf{c}}(D) \mid \mathbf{f}(D)$), or D is stepped ($\bar{\mathbf{c}}(D) \nmid \mathbf{f}(D)$) and $\mathbf{e}(D) = p \cdot \mathbf{f}(D)$. Moreover, the corollary shows that every TR subfield is contained in a SR subalgebra if and only if every TR subfield is **maximal** in some SR subalgebra.

Finally, an observation regarding the number of distinct subalgebras of a decomposable F-division algebra D.

COROLLARY 3.10: *If an F-division algebra D of p-power index has a proper nontrivial F-division subalgebra, then it has infinitely many nonisomorphic F-division subalgebras, ranging over all indexes dividing the index of D.*

Proof: In the proof of Lemma 3.4(i), the Grunwald–Wang Theorem was used to prove the existence of a character χ_1 with a (finite) set of prescribed local orders. By including additional irrelevant primes in this set, it is possible to produce an arbitrarily large number of **distinct** characters χ_1 that all satisfy (3.4.1) (but differ at the irrelevant primes). This modification to the proof shows there are infinitely many distinct characters such that the SR F-division algebras $D_1 = \Delta(\chi_1, t)$ are subalgebras of D. The injection $\mathbf{X}(k) \hookrightarrow \mathrm{Br}(F)$ shows that these distinct characters produce distinct Brauer elements (χ_1, t), hence nonisomorphic D_1.

The semiramified factor δ_1 described in Theorem 3.3(i) assume all values dividing $\underline{\mathbf{c}}(\delta)$. Hence δ_1 and δ_2 together assume all values dividing $\mathbf{i}(\delta)$. ∎

4. The reciprocal problem: embeddability

As in the last section, let k be a number field, and let F be a rank one discretely Henselian field with residue field k. If D is an F-division algebra, then D is decomposable if and only if it has a proper F-division subalgebra, as indicated in the discussion on decomposability in §1. Thus all of the results so far can be interpreted as results on the existence and structure of the division subalgebras of a given F-division algebra. The results in this section concern the reciprocal problem of existence and structure of the F-division algebras that (properly) contain a given F-division algebra.

By Theorem 2.2, there are at most 2 nontrivial factors in any decomposition of a given F-division algebra D of p-power index. Thus any proper F-division subalgebra is maximal, and if D is already decomposable, then it cannot properly embed in an F-division algebra of p-power index. If D is indecomposable, such an embedding might exist. In this section it is shown that "usually" it does. First it is shown that all semiramified F-division algebras D embed with all of the latitude allowed by the final statement of Theorem 3.3, and Remark 3.5.3(i). This result compliments the result proved earlier that a decomposable F-division algebra (of p-power index) always contains a semiramified D (Theorem 2.2). Then the main theorem of the section is proved, that all indecomposable F-division algebras of p-power index embed properly in some F-division algebra of p-power index if and only if there holds a proviso involving the Special Case of the Grunwald–Wang Theorem. (See §1 for background on the Special Case.)

As usual, Brauer elements will be used to analyze the problem, which then has the form: Given an element $\varepsilon \in \mathrm{Br}(F)$, does there exist an element δ such that $\mathbf{i}(\delta) = \mathbf{i}(\varepsilon)\,\mathbf{i}(\delta - \varepsilon)$?

THEOREM 4.1: *Let $\psi \in \mathbf{X}(F)$ be unramified and nontrivial of p-power order. Then there exists an element $\delta \in \mathrm{Br}(F)$ of p-power index such that $\Delta(\delta) \cong \Delta(\psi, t) \otimes \Delta(\delta - (\psi, t))$ is a nontrivial decomposition. Such δ exist with all indexes strictly divisible by $|\psi|$, and with all ramification and residue index combinations allowed by Theorem 3.3. That is, with either $\mathbf{e}(\delta) = |\psi|$ and $\mathbf{f}(\delta)$ arbitrary (but nontrivial), or $\mathbf{e}(\delta)$ and $\mathbf{f}(\delta)$ both divisible by $|\psi|$.*

Proof: Suppose $\Delta(\delta) \cong \Delta(\psi, t) \otimes \Delta(\delta - (\psi, t))$ is a decomposition. If (ψ, t) assumes the role of δ_2 in Theorem 3.3(i) or (ii), then $\mathbf{e}(\delta) = |\psi|$ (since $\mathbf{e}(\delta) = \mathbf{e}(\delta_2)$), and there are no a priori restrictions on $\mathbf{f}(\delta)$ except that it be nontrivial. If (ψ, t) assumes the role of δ_1 in Theorem 3.3, then necessarily $|\psi| \,|\, \mathbf{e}(\delta)$ (since $\mathbf{e}(\delta_1) \,|\, \mathbf{e}(\delta_2) = \mathbf{e}(\delta)$), and $|\psi| \,|\, \mathbf{f}(\delta)$ (since $|\psi| \,|\, \underline{\mathbf{c}}(\delta)$ and $\underline{\mathbf{c}}(\delta) \,|\, \mathbf{f}(\delta)$ by Theorem 1.3(vi)). Thus the ramification and residue index combinations allowed by Theorem 3.3 are either $\mathbf{e}(\delta) = |\psi|$ and $\mathbf{f}(\delta)$ arbitrary (but nontrivial), or $\mathbf{e}(\delta), \mathbf{f}(\delta)$ any numbers both divisible by $|\psi|$.

It is simple to construct a δ in the first case: Let α be unramified of index any number $\mathbf{f}(\delta)$ divisible by p, and with ramification locus disjoint from the locus of ψ. This is possible by Cebotarev's Density Theorem ([Ne]), which says there exist infinitely many primes not in the locus of ψ, and by Hasse's Theorem. Let $\delta = \alpha \dotplus (\psi, t)$. Then $\mathbf{i}(\alpha^{K(\psi)}) = \mathbf{f}(\delta) = \mathbf{i}(\alpha)$ by the local-global index formula, hence $\mathbf{i}(\delta) = \mathbf{i}(\alpha)\,\mathbf{i}((\psi, t))$ by Index Formula 1.1. Therefore $\Delta(\delta) \cong \Delta(\alpha) \otimes \Delta(\psi, t)$ is a (split) decomposition, of ramification index $\mathbf{e}(\delta) = |\psi|$ and residue index $\mathbf{f}(\delta)$ any number divisible by p.

The remaining case will follow from the following slightly more general result, which will be used in §5 to produce examples of decompositions.

LEMMA 4.2: *Let ψ be a nontrivial character of order n a p-power. Select p-power numbers*

$$\begin{aligned} e,\ f &: \quad n \,|\, (e, f) \qquad\qquad a : \quad f \,|\, a \,|\, \frac{ef}{n} \\ e_{\mathfrak{p}_1} = e_{\mathfrak{p}_2} &: \quad \frac{an}{f} \,\Big|\, e_{\mathfrak{p}_i} \,|\, e \quad (i = 1, 2) \end{aligned} \tag{4.2.1}$$

Then there exists an element $\delta = \alpha \dotplus (\chi, t) \in \mathrm{Br}(F)$ of p-power index, with nontrivial decomposition

$$\Delta(\delta) \cong \Delta(\psi, t) \otimes \Delta(\delta - (\psi, t))$$

such that $\mathbf{e}(\delta) = \mathbf{e}(\delta - (\psi, t)) = e$, $\mathbf{f}(\delta) = f$, $\mathbf{i}(\alpha) = a$, and $\mathbf{e}(\delta_{\mathfrak{p}_i}) = e_{\mathfrak{p}_i}$.

Proof: By Cebotarev's Density Theorem, there exist two primes $\mathfrak{q}_1$ and $\mathfrak{q}_2$ at which ψ has maximum order n, and two other primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ not dividing 2 where ψ is split completely. Let $\alpha \in \mathrm{Br}(k)$ have ramification locus $\{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{p}_1, \mathfrak{p}_2\}$, index f and opposing invariants at $\mathfrak{q}_1$ and $\mathfrak{q}_2$, and index a and opposing invariants at $\mathfrak{p}_1$ and $\mathfrak{p}_2$. This is allowed by Hasse's Theorem. Since $f \,|\, a$ by (4.2.1), $\mathbf{i}(\alpha) = a$, as desired. Now let $\chi_{\mathfrak{q}_i}$ $(i = 1, 2)$ be two trivial characters, and let $\chi_{\mathfrak{p}_i}$ $(i = 1, 2)$ be two characters of order $e_{\mathfrak{p}_i}$. Since $e_{\mathfrak{p}_i}$ divides e (by (4.2.1)), there exists by the Grunwald–Wang Theorem a character χ of order e with the $\chi_{\mathfrak{p}_i}$ and $\chi_{\mathfrak{q}_i}$ as completions (there is no Special Case since the primes $\mathfrak{p}_i$ do not divide 2).

Now let $\delta = \alpha \dotplus (\chi, t)$. Then $\mathbf{e}(\delta) = e$ and $\mathbf{e}(\delta_{\mathfrak{p}_i}) = e_{\mathfrak{p}_i}$, as desired.

CLAIM: $\Delta(\delta) \cong \Delta(\psi, t) \otimes \Delta(\delta - (\chi, t))$ is a nontrivial decomposition. As usual, it suffices to prove

$$\mathbf{i}(\delta) = \mathbf{i}((\psi, t)) \cdot \mathbf{i}(\delta - (\chi, t)). \tag{4.2.2}$$

The character of $\delta - (\psi, t)$ is $\chi - \psi$, and the unramified part with respect to t is α.

Since $|\chi| = e$ and $|\psi| = n$, $|\chi| \geq |\psi|$ by (4.2.1). If $|\chi| > |\psi|$, then $|\chi - \psi| \equiv \mathbf{e}(\delta - (\chi, t)) = e$. If $|\chi| = |\psi|$, then since χ is trivial and ψ has full order n at the $\mathfrak{q}_i$, $(\chi - \psi)_{\mathfrak{q}_i} = -\psi_{\mathfrak{q}_i}$, and both have order e. Hence in any case, $\mathbf{e}(\delta - (\chi, t)) = e$, as desired. Therefore by (2.0.1), to prove (4.2.2) and the rest of the lemma it remains to show $\mathbf{f}(\delta) \equiv \mathbf{i}(\alpha^{K\langle\chi\rangle}) = f$, and that $f = n \cdot \mathbf{f}(\delta - (\chi, t))$.

By construction, $\mathbf{i}(\alpha_{\mathfrak{q}_i}^{K\langle\chi_{\mathfrak{q}_i}\rangle}) = f$, and by local class field theory, $\mathbf{i}(\alpha_{\mathfrak{p}_i}^{K\langle\chi_{\mathfrak{p}_i}\rangle}) = \max\{1, a/e_{\mathfrak{p}_i}\}$. Since $(a/f) \,|\, e_{\mathfrak{p}_i}$ (by (4.2.1)), $(a/e_{\mathfrak{p}_i}) \,|\, f$. Therefore by the local-global index formula, $\mathbf{i}(\alpha^{K\langle\chi\rangle}) = f$. It remains to show $\mathbf{f}(\delta - (\chi, t)) = f/n$. By construction, $\chi_{\mathfrak{q}_i} - \psi_{\mathfrak{q}_i} = \psi_{\mathfrak{q}_i}$, and $\chi_{\mathfrak{p}_i} - \psi_{\mathfrak{p}_i} = \chi_{\mathfrak{p}_i}$, so $\mathbf{i}(\alpha^{K\langle\chi-\psi\rangle}) = \max\{\mathbf{i}(\alpha_{\mathfrak{p}_i}^{K\langle\chi_{\mathfrak{p}_i}\rangle}), \mathbf{i}(\alpha_{\mathfrak{q}_i}^{K\langle\psi_{\mathfrak{q}_i}\rangle})\}$. By construction this is $\max\{a/e_{\mathfrak{p}_i}, f/n, 1\}$. But by (4.2.1), $a/e_{\mathfrak{p}_i} \,|\, f/n$ and $1 \leq f/n$. Therefore $\mathbf{i}(\alpha^{K\langle\chi-\psi\rangle}) \equiv \mathbf{f}(\delta - (\chi, t)) = f/n$. This proves the claim, hence the lemma. ∎

CONTINUE PROOF OF THEOREM 4.1. It remains to show that the construction in Lemma 4.2 will yield a δ with any $\mathbf{e}(\delta)$ and $\mathbf{f}(\delta)$ both divisible by $|\psi| = n$. But this is immediate, since the δ constructed was shown to have invariants $\mathbf{e}(\delta) = e$ and $\mathbf{f}(\delta) = f$, and e and f were selected arbitrarily such that $n \,|\, (e, f)$ by (4.2.1). This completes the proof. ∎

Remarks 4.2.3: (i) The theorem supports an analogy between F-field extensions and F-division algebras: Just as a totally ramified F-field extension of degree n embeds in F-field extensions of all degrees divisible by n, a semiramified F-division algebra of index n embeds in F-division algebras of all indexes divisible by n. On the other hand, as pointed out in Remark 3.5.3(i), the larger F-field extension can have any ramification index divisible by n, and any residue index, whereas Theorem 3.3 puts stricter limitations on the corresponding invariants of the larger division algebra.

(ii) By Theorem 2.2, every decomposable F-division algebra (of p-power index) has a proper semiramified subalgebra. By the above, every semiramified F-division algebra of p-power index is a proper subalgebra of an F-division algebra of p-power index.

(iii) In §5 the parameters $a, f, e_{\mathfrak{p}_i}$, and e of Lemma 4.2 will be manipulated to produce examples of decompositions.

The main theorem of the section is next.

THEOREM 4.3: *Every indecomposable F-division algebra of p-power index embeds properly in some F-division algebra of p-power index if and only if for every prime $\mathfrak{q}$ of k, every local character $\psi_{\mathfrak{q}} \in \mathbf{X}(k_{\mathfrak{q}})$ has a Grunwald lift. In particular, if p is odd, or if $p = 2$ and k has no Special Case, then every indecomposable F-division algebra properly embeds.*

Proof: Suppose every local character has a Grunwald lift. Let $\Delta(\delta') \cong \Delta(\alpha \dotplus (\chi', t))$ be indecomposable, for some unramified α and character χ', all of p-power order. To prove the reverse implication it suffices to construct a $\delta \in \mathrm{Br}(F)$ such that $\mathbf{i}(\delta) = \mathbf{i}(\delta - \delta') \cdot \mathbf{i}(\delta')$. This has already been accomplished if δ' is SR (in Theorem 4.1), so assume that δ' is not SR.

If δ' is unramified, so $\delta' = \alpha$, let χ be any nontrivial character that is trivial at all primes $\mathfrak{q}$ at which α takes maximal index. By the Grunwald–Wang Theorem, such a character exists, of any order divisible by p. Let $\delta = \alpha \dotplus (\chi, t)$. Then $\mathbf{i}(\delta) = \mathbf{i}(\alpha^{K(\chi)}) \cdot \mathbf{i}((\chi, t)) = \mathbf{i}(\alpha) \cdot \mathbf{i}((\chi, t))$, so $\Delta(\delta) \cong \Delta(\alpha) \otimes \Delta(\chi, u)$ is a (split) decomposition, with $\Delta(\delta') = \Delta(\alpha)$ a factor. (This is the **only** way an unramified element may appear as a factor in a decomposition, since by Theorem 2.2 the complimentary factor must be SR.)

Now assume δ' is neither unramified nor SR. Thus $p \,|\, (\mathbf{f}(\delta'), \mathbf{e}(\delta'))$. Since $\Delta(\delta')$ is indecomposable, by Corollary 3.6 there exists a prime $\mathfrak{q}$ of k at which $\mathbf{f}(\delta') = \mathbf{f}(\delta'_{\mathfrak{q}})$ and $p \,|\, \mathbf{e}(\delta'_{\mathfrak{q}})$. Set $\psi_{\mathfrak{q}} = -\chi'_{\mathfrak{q}}$ at this $\mathfrak{q}$. By hypothesis $\psi_{\mathfrak{q}}$ has a Grunwald lift, so let ψ be a character of order $\mathbf{e}(\delta'_{\mathfrak{q}})$ and with completion $\psi_{\mathfrak{q}}$ at $\mathfrak{q}$, and with trivial completion at some other prime where χ' has full order $\mathbf{e}(\delta')$. Such a prime exists by Cebotarev's Density Theorem ([Ne]), which shows there are infinitely many primes at which χ' has full order. Let $\chi = \psi + \chi'$, and $\delta = \alpha \dotplus (\chi, t)$.

CLAIM: $\Delta(\delta) \cong \Delta(\delta - \delta') \otimes \Delta(\delta')$ is a nontrivial decomposition, i.e., $\mathbf{i}(\delta) = \mathbf{i}(\delta - \delta') \cdot \mathbf{i}(\delta')$. Here, $\delta - \delta' = (\psi, t)$. Since $|\psi| = \mathbf{e}(\delta'_{\mathfrak{q}}) \leq |\chi'|$ and ψ is trivial at a prime where χ' has full order, $|\chi| = |\psi + \chi'| = |\chi'|$. Therefore it suffices to prove $\mathbf{f}(\delta) = |\psi| \cdot \mathbf{f}(\delta')$, by (2.0.1).

By construction, $\chi_{\mathfrak{q}} = 0$ at the chosen prime $\mathfrak{q}$, hence $\mathbf{i}(\alpha_{\mathfrak{q}}^{K\langle\chi_{\mathfrak{q}}\rangle}) = \mathbf{i}(\alpha_{\mathfrak{q}})$. Since $\mathbf{f}(\delta') = \mathbf{f}(\delta'_{\mathfrak{q}})$ (by choice of $\mathfrak{q}$) and $\mathbf{f}(\delta') \neq 1$ (by hypothesis), the formula for residue index shows that $K\langle\chi'_{\mathfrak{q}}\rangle$ does not split $\alpha_{\mathfrak{q}}$, hence by local class field theory, $\mathbf{i}(\alpha_{\mathfrak{q}}) = |\chi'_{\mathfrak{q}}| \cdot \mathbf{i}(\alpha_{\mathfrak{q}}^{K\langle\chi'_{\mathfrak{q}}\rangle})$. Thus, since $|\chi'_{\mathfrak{q}}| = |\psi|$, $\mathbf{i}(\alpha_{\mathfrak{q}}^{K\langle\chi'_{\mathfrak{q}}\rangle}) = |\psi| \cdot \mathbf{f}(\delta')$. If $\mathbf{i}(\alpha_{\mathfrak{p}}^{K\langle\chi_{\mathfrak{p}}\rangle}) \leq |\psi| \cdot \mathbf{f}(\delta')$ at all $\mathfrak{p} \neq \mathfrak{q}$, then the claim is proved by the local-global index formula. But if $|\chi_{\mathfrak{p}}| \geq |\chi'_{\mathfrak{p}}|$, then $\mathbf{i}(\alpha_{\mathfrak{p}}^{K\langle\chi_{\mathfrak{p}}\rangle}) \leq \mathbf{i}(\alpha_{\mathfrak{p}}^{K\langle\chi'_{\mathfrak{p}}\rangle}) \leq \mathbf{f}(\delta')$. If $|\chi_{\mathfrak{p}}| < |\chi'_{\mathfrak{p}}|$, then since $\chi_{\mathfrak{p}} = \psi_{\mathfrak{p}} + \chi'_{\mathfrak{p}}$, $|\chi'_{\mathfrak{p}}| = |\psi_{\mathfrak{p}}|$. Since $\mathbf{i}(\alpha_{\mathfrak{p}}^{K\langle\chi'_{\mathfrak{p}}\rangle}) \leq \mathbf{f}(\delta')$, necessarily $\mathbf{i}(\alpha_{\mathfrak{p}}) \leq |\chi'_{\mathfrak{p}}| \cdot \mathbf{f}(\delta')$, hence $\mathbf{i}(\alpha_{\mathfrak{p}}^{K\langle\chi_{\mathfrak{p}}\rangle}) \leq |\chi'_{\mathfrak{p}}| \cdot \mathbf{f}(\delta')$, hence $\mathbf{i}(\alpha_{\mathfrak{p}}^{K\langle\chi_{\mathfrak{p}}\rangle}) \leq |\psi_{\mathfrak{p}}| \cdot \mathbf{f}(\delta') \leq |\psi| \cdot \mathbf{f}(\delta')$. This establishes the claim, and completes the proof of the reverse implication of the theorem.

If k has a Special Case, then the above construction of the element δ **snags** when $-\chi'_{\mathfrak{q}}$ has no Grunwald lift. The forward implication of the theorem consists of a demonstration that things can be arranged so that no remedy is possible. The counterexample δ' will necessarily be of 2-power index, with character χ' of order divisible by 2^{s+1}, where s is as in the discussion of the Grunwald–Wang Theorem in §1.

Let $\mathfrak{q}$ be a prime and let $\chi'_{\mathfrak{q}}$ be a character of order m a 2-power such that $-\chi'_{\mathfrak{q}}$ has no Grunwald lift (necessarily $8 \mid 2^{s+1} \mid m$). Let $\mathfrak{p}$ be a prime not dividing 2, and let $\chi'_{\mathfrak{p}}$ be a character of order $2m$. Let χ' be a (global) character of order $2m$ with $\chi'_{\mathfrak{q}}$ and $\chi'_{\mathfrak{p}}$ as completions. This exists by Grunwald–Wang. Let f' be a 2-power divisible by m (hence by 8). By Hasse's Theorem, there exists an element $\alpha \in \mathrm{Br}(k)$ of index mf' whose ramification locus consists of $\mathfrak{p}$ and $\mathfrak{q}$. Of course, then $\mathrm{inv}_{\mathfrak{q}}(\alpha) \equiv -\mathrm{inv}_{\mathfrak{p}}(\alpha)$, where inv is the invariant map ([Re] Ch. 32).

Let $\delta' = \alpha \dotplus (\chi', t)$. Then $\mathbf{e}(\delta') = 2m$, and $\mathbf{f}(\delta') = \mathbf{i}(\alpha^{K\langle\chi'\rangle}) = \mathbf{i}(\alpha_{\mathfrak{q}}^{K\langle\chi'_{\mathfrak{q}}\rangle}) = mf'/m = f'$. Since $\mathbf{e}(\delta'_{\mathfrak{q}}) \neq 1$ and $\mathbf{f}(\delta'_{\mathfrak{q}}) = \mathbf{f}(\delta')$, $\Delta(\delta')$ is indecomposable by Corollary 3.6.

CLAIM: $\Delta(\delta')$ cannot be a part of any decomposition. This result will prove the theorem.

Suppose otherwise, i.e., that $\Delta(\delta) \cong \Delta(\delta - \delta') \otimes \Delta(\delta')$ is a decomposition. Let χ be the character of δ. Since δ' is not SR ($\mathbf{f}(\delta') = f'$ is divisible by 8), $\delta - \delta'$ must be

SR by Theorem 2.2. Therefore $\delta - \delta' = (\psi, t/u)$, for $\psi := \chi - \chi'$ and some $u \in k^{\cdot}$. Since $\delta' = \alpha + (\chi', u) \dotplus (\chi', t/u)$, solving for δ yields $\delta = \alpha + (\chi', u) \dotplus (\chi, t/u)$.

If $\mathbf{e}(\delta - \delta') \geq \mathbf{e}(\delta')$, then by Lemma 2.1 both $\mathbf{e}(\delta')$ and $\mathbf{e}(\delta - \delta')$ divide $\mathbf{e}(\delta)$, and $\mathbf{e}(\delta - \delta') = \mathbf{e}(\delta)$. Since $\delta - \delta'$ is SR, this means there exists a TR subfield of $\Delta(\delta)$ of degree $\mathbf{e}(\delta)$, hence $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$ by Definition 1.2. Thus by Theorem 2.2(ii), with δ' assuming the role of δ_1, $\mathbf{f}(\delta) = \mathbf{i}(\delta') = \mathbf{o}(\delta')$. Since δ' is not SR $(8 \mid f')$, $\mathbf{i}(\delta') = \mathbf{i}(\alpha + (\chi', u))$ by Lemma 1.4. But this equation is false: By Index Formula 1.1, $\mathbf{i}(\delta') = |\chi'| \cdot \mathbf{f}(\delta') = 2mf'$, while by construction, $\mathbf{i}(\alpha) = mf'$. Since $|\chi'| = 2m$, and $8 \mid f'$, $|\chi'| < \mathbf{i}(\alpha)$, hence, since k is absolutely stable, $\mathbf{i}(\alpha + (\chi', u)) = \mathbf{i}(\alpha)$. Hence $\mathbf{i}(\alpha + (\chi', u)) = mf'$. Since $2mf' \neq mf'$, this is a contradiction. The conclusion is $\mathbf{e}(\delta - \delta') < \mathbf{e}(\delta') = \mathbf{e}(\delta)$.

Since $\mathbf{e}(\delta) = \mathbf{e}(\delta')$ and $\delta - \delta'$ is SR, $\mathbf{f}(\delta) = \mathbf{e}(\delta - \delta')\mathbf{f}(\delta')$ by (2.0.1). Since χ' has order $\mathbf{e}(\delta') = 2m$, and $\mathbf{e}(\delta - \delta') < \mathbf{e}(\delta')$, $\mathbf{e}(\delta - \delta') \mid m$. Since $\Delta(\delta) \cong \Delta(\alpha + (\chi', u) \dotplus (\chi, t/u))$ is decomposable by hypothesis, by Corollary 3.6 there exists a prime $\mathfrak{q}' \in \mathrm{Loc}(\alpha + (\chi', u))$ at which $\mathbf{f}(\delta_{\mathfrak{q}'}) = \mathbf{f}(\delta)$, and $\mathbf{e}(\delta_{\mathfrak{q}'}) \equiv |\chi_{\mathfrak{q}'}| = 1$.

Since $|\chi_{\mathfrak{q}'}| = 1$ and $\chi = \psi + \chi'$, $|\psi_{\mathfrak{q}'}| = |\chi'_{\mathfrak{q}'}|$. Since $|\psi_{\mathfrak{q}'}| \mid |\psi| \equiv \mathbf{e}(\delta - \delta')$ and $\mathbf{e}(\delta - \delta') = \mathbf{f}(\delta)/\mathbf{f}(\delta') < \mathbf{f}(\delta)$ $(8 \mid \mathbf{f}(\delta'))$, necessarily $|\psi_{\mathfrak{q}'}| < \mathbf{f}(\delta)$, hence $|\chi'_{\mathfrak{q}'}| < \mathbf{f}(\delta)$, hence $\mathbf{i}((\chi', u)_{\mathfrak{q}'}) < \mathbf{f}(\delta)$. The equalities $\mathbf{f}(\delta_{\mathfrak{q}'}) = \mathbf{f}(\delta)$ and $|\chi_{\mathfrak{q}'}| = 1$ together imply $\mathbf{i}((\alpha + (\chi', u))_{\mathfrak{q}'}) = \mathbf{f}(\delta)$. Since $\mathbf{i}((\chi', u)_{\mathfrak{q}'}) < \mathbf{f}(\delta)$ and $k_{\mathfrak{q}'}$ is absolutely stable, this forces $\mathbf{i}(\alpha_{\mathfrak{q}'}) = \mathbf{f}(\delta)$. Consequently, $\mathfrak{q}' \in \mathrm{Loc}(\alpha)$.

Since $\mathfrak{q}' \in \mathrm{Loc}(\alpha)$, it is either $\mathfrak{p}$ or $\mathfrak{q}$. But it can't be $\mathfrak{p}$: For $|\chi'_{\mathfrak{p}}| = |\chi'|$ by construction, and as already shown, $|\chi'| > |\psi|$. Therefore $|\chi'_{\mathfrak{p}}| > |\psi_{\mathfrak{p}}|$, hence $|\chi'_{\mathfrak{p}}| = |\chi_{\mathfrak{p}}|$, and in particular, $|\chi_{\mathfrak{p}}| \neq 1$. Therefore χ must be split completely at $\mathfrak{q}$, and so $-\chi'_{\mathfrak{q}} = \psi_{\mathfrak{q}}$. By construction, $|-\chi'_{\mathfrak{q}}| = m$, so $|\psi_{\mathfrak{q}}| = m$. Always $|\psi_{\mathfrak{q}}| \mid |\psi|$, and by the above, $|\psi| \equiv \mathbf{e}(\delta - \delta') \mid m$, so this forces $|\psi| = m$. But then ψ is a Grunwald lift of $\psi_{\mathfrak{q}} = -\chi'_{\mathfrak{q}}$. By hypothesis, no such lift exists! This proves the claim, and completes the proof of the theorem. ∎

COROLLARY 4.4: *For certain number fields k there exist indecomposable F-division algebras of 2-power index that do not properly embed in any F-division algebras of 2-power index.*

Proof: By the theorem, it suffices to find a field with $|S_0| = 1$. For example, if $k = \mathbb{Q}$, the unramified character of order 8 at the prime 2 has no Grunwald lift ([AT], Ch. X). ∎

Remark 4.4.1: It should be possible to produce a local criterion that decides exactly when a given indecomposable F-division algebra of 2-power index divisible by 2^{s+1} embeds properly in a F-division algebra of 2-power index.

5. Examples of decompositions

In this section, the construction of Lemma 4.2 is manipulated to produce some examples of decompositions with interesting features. These occur over the rank one discretely Henselian field F, with residue field k a number field.

LEMMA 5.1: *Let ψ be a nontrivial character of order n a p-power, and Let δ be as constructed in Lemma 4.2, with $f = \mathbf{f}(\delta)$, $a = \mathbf{i}(\alpha)$, $e_{\mathfrak{p}_i} = \mathbf{e}(\delta_{\mathfrak{p}_i})$, and $e = \mathbf{e}(\delta)$. Let $\Delta(\delta) \cong \Delta(\psi, t) \otimes \Delta(\delta - (\psi, t))$ be the decomposition of Lemma 4.2. Then*

(i) *$\delta - (\psi, t)$ is semiramified if and only if $n = f$.*

(ii) *$f \mid \underline{\mathbf{c}}(\delta)$ if and only if $f = a \mid e$ or $a \mid e_{\mathfrak{p}_i}$.*

(iii) *$\Delta(\delta)$ has a split decomposition if and only if $f = a$ or $a \mid e_{\mathfrak{p}_i}$.*

Proof: The factor $\delta - (\psi, t)$ in Lemma 4.2 is SR if and only if $\mathbf{i}(\delta - (\psi, t)) = \mathbf{e}(\delta - (\psi, t))$. Since $\mathbf{i}(\delta) = \mathbf{i}((\psi, t)) \cdot \mathbf{i}(\delta - (\psi, t))$ and $\mathbf{e}(\delta - (\psi, t)) = \mathbf{e}(\delta)$, $\delta - (\psi, t)$ is SR if and only if $\mathbf{f}(\delta) = \mathbf{i}((\psi, t))$, i.e., $f = n$. This proves (i).

By Theorem 1.3(vi), $f \mid \underline{\mathbf{c}}(\delta)$ if and only if $\mathbf{f}(\delta) = \underline{\mathbf{c}}(\delta)$. By Theorem 1.3(v), to show $\mathbf{f}(\delta) = \underline{\mathbf{c}}(\delta)$ is equivalent to showing $\mathbf{f}(\delta) \mid c_1(\delta, t)$ (see Definition 1.2 for the definition of $c_u(\delta, t)$).

Let $T^u = \mathrm{Loc}(\mathbf{f}(\delta) \cdot (\alpha + (\chi, 1/u)))$, $u \in k^{\cdot}$, where δ is as in Lemma 4.2. Thus T^u is the locus of primes at which $\alpha + (\chi, 1/u)$ has local index greater than the global residue index $\mathbf{f}(\delta)$. By Theorem 5.12, [Br2], either $T^u = \varnothing$, in which case $c_u(\delta, t) = \bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$, or else $T^u \neq \varnothing$, and then

$$c_u(\delta, t) = \min_{T^u} \left\{ \frac{\mathbf{f}(\delta) \cdot \mathbf{e}(\delta_{\mathfrak{q}})}{\mathbf{i}((\alpha + (\chi, 1/u))_{\mathfrak{q}})} \right\}. \tag{5.1.1}$$

If $u = 1$, then $T^u = \mathrm{Loc}(\mathbf{f}(\delta) \cdot \alpha)$. Thus if $\mathbf{f}(\delta) = \mathbf{i}(\alpha)$ in Lemma 4.2, then $T^1 = \varnothing$, and $f \mid \underline{\mathbf{c}}(\delta)$ if and only if $\mathbf{f}(\delta) \mid \mathbf{e}(\delta)$, i.e., $f (= a) \mid e$.

If $\mathbf{f}(\delta) < \mathbf{i}(\alpha)$, then $T^1 = \{\mathfrak{p}_1, \mathfrak{p}_2\}$: For by construction, $\mathrm{Loc}(\alpha) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2\}$, $\mathbf{i}(\alpha_{\mathfrak{p}_i}) = \mathbf{i}(\alpha)$, and $\mathbf{i}(\alpha_{\mathfrak{q}_i}) = \mathbf{f}(\delta)$. Hence $\mathbf{i}(\mathbf{f}(\delta) \cdot \alpha_{\mathfrak{p}_i}) = \mathbf{i}(\alpha)/\mathbf{f}(\delta)$ and $\mathbf{i}(\mathbf{f}(\delta) \cdot \alpha_{\mathfrak{q}_i}) = 1$, hence $\mathrm{Loc}(\mathbf{f}(\delta) \cdot \alpha) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. In this case, (5.1.1) becomes $c_1(\delta, t) = \mathbf{f}(\delta) \cdot \mathbf{e}(\delta_{\mathfrak{p}_i}) / \mathbf{i}(\alpha)$. Therefore when $f < a$, $f \mid \underline{\mathbf{c}}(\delta)$ if and only if $a \mid e_{\mathfrak{p}_i}$.

If $\mathbf{i}(\alpha) \mid \mathbf{e}(\delta_{\mathfrak{p}_i})$, then $\mathbf{f}(\delta) \mid \mathbf{e}(\delta)$, since always $\mathbf{f}(\delta) \mid \mathbf{i}(\alpha)$ and $e(\delta_{\mathfrak{p}_i}) \mid \mathbf{e}(\delta)$. Therefore $a \mid e_{\mathfrak{p}_i}$ implies $f \mid \underline{\mathbf{c}}(\delta)$ in any case. This completes the proof of (ii).

By Lemma 1.4, $\Delta(\delta)$ has a split decomposition if and only if $\mathbf{f}(\delta) \neq 1$ and $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$. In Lemma 4.2, $p \mid n \mid \mathbf{f}(\delta)$, so $\mathbf{f}(\delta) \neq 1$. If $\mathbf{f}(\delta) \mid \underline{\mathbf{c}}(\delta)$, then by Theorem 1.3(vi), $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$, hence $\Delta(\delta)$ has a split decomposition in this case. If δ is stubbed, then $\underline{\mathbf{c}}(\delta) = \bar{\mathbf{c}}(\delta)$ by Theorem 1.3(v), so $\bar{\mathbf{c}}(\delta) = \mathbf{e}(\delta)$ if and only if $c_1(\delta, t) = \mathbf{e}(\delta)$. If $T^1 = \varnothing$, i.e., $f(\delta) = \mathbf{i}(\alpha)$, then $c_1(\delta, t) = \mathbf{e}(\delta)$ by the above, so there is a split decomposition in this case. If $T^1 \neq \varnothing$, so $\mathbf{f}(\delta) < \mathbf{i}(\alpha)$, then by the computation above, $c_1(\delta, t) = \mathbf{f}(\delta)\mathbf{e}(\delta_{\mathfrak{p}_i})/\mathbf{i}(\alpha)$. Since $\mathbf{f}(\delta) < \mathbf{i}(\alpha)$ and $\mathbf{e}(\delta_{\mathfrak{p}_i}) \mid \mathbf{e}(\delta)$ by (4.2.1), $c_1(\delta, t) = \mathbf{e}(\delta)$ is impossible. Conclude, using (ii), that if $\mathbf{f}(\delta) = \mathbf{i}(\alpha) \mid \mathbf{e}(\delta)$ or $\mathbf{i}(\alpha) \mid \mathbf{e}(\delta_{\mathfrak{p}_i})$ (the case $f \mid \underline{\mathbf{c}}(\delta)$) then $\Delta(\delta)$ has a split decomposition, and if $\mathbf{f}(\delta) = \mathbf{i}(\alpha) \nmid \mathbf{e}(\delta_{\mathfrak{p}_i})$ (the stubbed case) then $\Delta(\delta)$ has a split decomposition also. Therefore if $f = a$ or $a \mid e_{\mathfrak{p}_i}$, then $\Delta(\delta)$ of Lemma 4.2 has a split decomposition.

Conversely, suppose $f < a$ and $a \nmid e_{\mathfrak{p}_i}$. Then by (ii), δ is stubbed, hence $c_1(\delta, t) = \bar{\mathbf{c}}(\delta)$ by Theorem 1.3(v). As above, $T^1 \neq \varnothing$ and $c_1(\delta, t) = \mathbf{e}(\delta)$ is impossible. Therefore $\bar{\mathbf{c}}(\delta) \neq \mathbf{e}(\delta)$, hence $\Delta(\delta)$ does not have a split decomposition. This proves (iii), and finishes the proof of the lemma. ∎

Using this result, it is possible to manipulate the parameters in Lemma 4.2 to produce decompositions with various features. A few examples the author did not expect appear in the next corollary.

COROLLARY 5.2:

(i) *There exists an F-division algebra D with nontrivial decompositions*

$$D \cong A \otimes \Delta(\chi, t) \cong \Delta(\chi_1, t) \otimes \Delta(\chi_2, ut) \tag{5.2.1}$$

where A is unramified, χ, χ_1, and χ_2 are unramified characters, and $u \in k$. That is, D has a split decomposition into an unramified factor and a semiramified factor, and also a decomposition into 2 semiramified factors.

(ii) *There exists a stubbed F-division algebra D that has a split decomposition. Hence*

$$D \cong A \otimes \Delta(\chi, t), \qquad A \text{ unramified} \tag{5.2.2}$$

and D contains every totally ramified F-extension of degree dividing $\mathbf{e}(D) = |\chi|$.

(iii) *There exists an F-division algebra D that has a decomposition*

$$D \cong \Delta(\psi, t) \otimes D' \tag{5.2.3}$$

such that D' has no totally ramified subfield, and D has totally ramified subfields of degree strictly larger than those of the semiramified factor $\Delta(\psi, t)$. D can be either stepped or stubbed.

Proof: Let ψ be a nontrivial character of order n, and let $D = \Delta(A \otimes \Delta(\chi, t))$ be the division algebra representing the element δ constructed from ψ in Lemma 4.2. Set $n = |\psi|$, $e = \mathbf{e}(D)$, $f = \mathbf{f}(D)$, $a = \mathbf{i}(A)$, $e = \mathbf{e}(D)$, and $e_{p_i} = \mathbf{e}(\delta_{\mathfrak{p}_i})$, as in Lemma 4.2.

By Index Formula 1.1 and definitions, D has the split decomposition $A \otimes \Delta(\chi, t)$ if $\mathbf{i}(A) = \mathbf{f}(D)$. By Lemma 4.2, $D \cong \Delta(\psi, t) \otimes \Delta(\delta - (\psi, t))$. Since $\mathbf{i}((\psi, t)) \equiv n$, the second factor is SR if and only if $\mathbf{e}(\delta - (\psi, t)) = \mathbf{i}(\delta)/n$. By Lemma 4.2, $\mathbf{e}(\delta - (\psi, t)) = \mathbf{e}(\delta)$, so this occurs exactly when $\mathbf{f}(\delta) = n$, by Index Formula 1.1.

The condition $n = a = f$ does not violate (4.2.1), hence the construction of D with $n = \mathbf{i}(A) = \mathbf{f}(D)$ exists. Since $\Delta(\delta - (\psi, t))$ is SR, it has form $\Delta(\chi_2, ut)$ for some character χ_2 and some $u \in k^{\cdot}$, as indicated in §1. Setting $\psi = \chi_1$ yields the statement (5.2.1), and proves (i).

The condition $f = a > e_{\mathfrak{p}_i} = e$ does not violate (4.2.1), hence the construction $D \cong \Delta(\psi, t) \otimes \Delta(\delta - (\psi, t))$ exists with these parameters. Since $a \nmid e_{\mathfrak{p}_i}$, $f \nmid \underline{\mathbf{c}}(D)$ by Lemma 5.1(ii), hence D is stubbed by Theorem 1.3(vi). Since $f = a$, D has a split decomposition by Lemma 5.1(iii). By Theorem 1.3(v), if D is stubbed then D contains every TR F-extension of degree dividing $\underline{\mathbf{c}}(D)$, and $\underline{\mathbf{c}}(D) = \bar{\mathbf{c}}(D)$. Since D has a split decomposition, $\bar{\mathbf{c}}(D) = \mathbf{e}(D)$ by Lemma 1.4. This proves (ii).

By Corollary 3.8, $\Delta(\delta - (\psi, t))$ has a TR subfield if and only if it has a SR subalgebra. Since $\Delta(\delta - (\psi, t))$ is indecomposable (by Theorem 2.2), it has no proper division subalgebras, hence either it is SR itself, or it has no TR subfields. Thus by Lemma 5.1(i), $\Delta(\delta - (\psi, t))$ will not be SR if $n < \mathbf{f}(D)$. By Definition 1.2, D will have TR subfields of degree larger than n $(= \mathbf{i}((\psi, t)))$ if $n < \bar{\mathbf{c}}(D)$. Therefore to prove the decomposition described in (5.2.3) exists, it suffices to show the D of Lemma 4.2 exists with $n < \bar{\mathbf{c}}(D)$, and $n < \mathbf{f}(D)$.

The condition $n < f = a < e$ does not violate (4.2.1). Therefore by Lemma 4.2 there is a decomposable D with $n < \mathbf{f}(D)$. Since $f = a \mid e$, D is stepped by Lemma 5.1(ii), hence $\bar{\mathbf{c}}(D) = \mathbf{e}(D)$ by Theorem 1.3(vi), hence $n < \bar{\mathbf{c}}(D)$ (since $n < \mathbf{f}(D) \mid \bar{\mathbf{c}}(D)$), as desired. To produce a stubbed example, take $n < e < f = a$ and $e_{\mathfrak{p}_i} = e$ $(i = 1, 2)$. Again this does not violate (4.2.1), and so again by Lemma 4.2 there is a decomposable D with $n < \mathbf{f}(D)$. This time $a \nmid e$, and since $e = e_{\mathfrak{p}_i}$,

D is stubbed by Lemma 5.1(ii). Since $f = a$, D has a split decomposition (again by Lemma 5.1), hence $\mathbf{e}(D) = \bar{\mathbf{c}}(D)$ by Lemma 1.4. Therefore $n < \bar{\mathbf{c}}(D)$, as desired. This completes the proof. ∎

6. Results over k(t)

Let k be a number field, let $k(t)$ be the rational function field in one variable, and let $k((t))$ be the completion of $k(t)$ at t. By a theorem of Auslander–Brummer, Fadeev (see e.g. [FSS]), there is a split exact sequence

$$(6.0.1) \qquad 0 \longrightarrow C \longrightarrow \mathrm{Br}(k(t)) \underset{s}{\overset{\mathrm{res}}{\rightleftarrows}} \mathrm{Br}(k((t))) \longrightarrow 0.$$

The map res is restriction to $k((t))$, earlier denoted by $\cdot^{k((t))}$. The map s is a splitting of res. If $\alpha \in \mathrm{Br}(k)$ and $\chi \in \mathbf{X}(k)$, then the element $\alpha \dotplus (\chi, t)$ may be defined over $k(t)$ by identifying α with its restriction to $k(t)$ and letting (χ, t) be the usual cup product of χ and t, or alternatively the class of the cyclic crossed product defined by χ and t over $k(t)$. Since res is just extension of the ground field, clearly $\mathrm{res}(\alpha + (\chi, t))$ is the element $\alpha \dotplus (\chi, t)$, defined over $k((t))$. Thus s may be defined so that $s(\mathrm{res}(\alpha \dotplus (\chi, t))) = \alpha \dotplus (\chi, t)$. Then, by Lemma 4 in [Br1],

$$(6.0.2) \qquad \mathbf{i}(s(\delta)) = \mathbf{i}(\delta).$$

Given Index Formula (1.1) for δ, this can be proved by noting that always $\mathbf{i}(s(\delta)) = \mathbf{i}(\alpha \dotplus (\chi, t)) \leq [K(t) : k(t)] \cdot \mathbf{i}((\alpha \dotplus (\chi, t))^{K(t)}) = \mathbf{i}(\delta)$, where $K(t) := K\langle \chi^{k(t)} \rangle$, while on the other hand, always $\mathbf{i}(\delta) = \mathbf{i}(\mathrm{res}(s(\delta))) \leq \mathbf{i}(s(\delta))$, since index can only go down upon restriction of scalars. Thus $\mathbf{i}(s(\delta)) = \mathbf{i}(\delta)$.

Since s is an injection,

$$(6.0.3) \qquad \mathbf{o}(s(\delta)) = \mathbf{o}(\delta).$$

It is possible to prove some decomposability results over $k(t)$ as direct corollaries of the results over the discretely Henselian field $F = k((t))$, using (6.0.2) and (6.0.3). The results apply only to the elements in $\mathrm{Br}(k(t))$ that are in the image of s, i.e., to elements of the form $\alpha \dotplus (\chi, t)$. However, the ring automorphism $k(t) \to k(t)$ defined by $t \mapsto t - u$, $u \in k^{\cdot}$ allows t to be replaced by any linear prime $t - u$ without affecting the substance of any of the results. Thus the results really apply to all elements of $\mathrm{Br}(k(t))$ of the form $\delta = \alpha + (\chi, t - u)$, $u \in k^{\cdot}$.

By [Jn] Ch. II, §5, the polynomial $X^e - ut$, $u \in k^{\cdot}$ is irreducible over $k((t))$, hence also over the subfield $k(t)$. Let τ be a root. Then $k(t)(\tau) \cong k(\tau)$, and $k((t))(\tau) \cong k((\tau))$: For $k(\tau) \supset k(t)(\tau)$, since $t = \tau^e \cdot 1/u$, and clearly $k(\tau) \subset k(t)(\tau)$. For the second isomorphism, see [Jn]. The extension $k((\tau))/k((t))$ $\tau^e = ut$ is totally ramified of degree e, and all totally ramified $k((t))$-extensions of degree e have this form, as noted in §1 preceeding Definition 1.2.

LEMMA 6.1: *Let $k(\tau)/k(t)$ and $k((\tau))/k((t))$ be extensions defined by the (irreducible) polynomial $X^e - ut$, $u \in k^{\cdot}$, as above. Let D be a $k(t)$-division algebra representing an element in the image of s. That is, $[D] = s(\mathrm{res}([D]))$ in (6.0.1). Then $D \otimes k((t))$ is a division algebra, and*

$$k(\tau) \subset D \iff k((\tau)) \subset D \otimes k((t)).$$

Sketch of Proof: That $D \otimes k((t))$ is a division algebra follows directly from (6.0.2). The theorem of Auslander–Brummer, Fadeev (6.0.1) applies to $k(\tau)$, since $k(\tau)$ is rational over k. It can be shown that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(k(\tau)) & \underset{s}{\overset{\mathrm{res}}{\rightleftarrows}} & \mathrm{Br}(k((\tau))) \\ \uparrow & & \uparrow \\ \mathrm{Br}(k(t)) & \underset{s}{\overset{\mathrm{res}}{\rightleftarrows}} & \mathrm{Br}(k((t))) \end{array} \qquad (6.1.1)$$

This follows from the functoriality of restriction of scalars. Since the injections s preserve index, a diagram chase shows that the index of $[D]$ is lowered by e upon restriction to $k(\tau)$ if and only if the index of $\mathrm{res}([D])$ is lowered by e upon restriction to $k((\tau))$. Therefore $k(\tau) \subset D$ if and only if $k((\tau)) \subset D \otimes k((t))$. ∎

If $\delta \in \mathrm{Br}(k(t))$, the residue and ramification indexes of δ with respect to t are defined to be the corresponding invariants of $\mathrm{res}(\delta) \in \mathrm{Br}(k((t)))$.

Furthermore, for δ in the image of s, Definition 1.2 can be made equally well over $k(t)$:

Definition 6.2: Let $D/k(t)$ be central simple, with Brauer class δ. The **ceiling number** $c_u(D,t) \equiv c_u(\delta,t)$ for $u \in k^{\cdot}$ is the largest n such that $F(\sqrt[n]{ut}) \subset \Delta(D)$. The **upper ceiling number** $\bar{c}(D) \equiv \bar{c}(\delta)$ and the **lower ceiling number** $\underline{c}(D) \equiv \underline{c}(\delta)$ are the least common multiple and the greatest common divisor of the $c_u(D,t)$, respectively, taken over all $u \in k$. Call D and δ **stubbed** if $\bar{\mathbf{c}}(D) \mid \mathbf{f}(D)$, and **stepped** if $\bar{\mathbf{c}}(D) \nmid \mathbf{f}(D)$.

By Lemma 6.1, the resulting parameters $c_u(\delta,t)$, $\underline{\mathbf{c}}(\delta)$, and $\bar{\mathbf{c}}(\delta)$ are equal to $c_u(\mathrm{res}(\delta),t)$, $\underline{\mathbf{c}}(\mathrm{res}(\delta))$, and $\bar{\mathbf{c}}(\mathrm{res}(\delta))$.

If $\delta \in \mathrm{Im}(s)$, then a decomposition for $\Delta(\delta)$ may involve nontrivial terms of C, which are **not** in the image of s. Thus a priori it is not clear that an element in the image of s that is indecomposable when restricted to $k((t))$ is indecomposable over $k(t)$. However, this turns out not to be a problem:

Proposition 6.3: *Suppose $\delta \in \mathrm{Br}(k(t))$ has p-power index, and is in the image of s in (6.0.1). Then*

$$\Delta(\delta) \text{ is decomposable} \iff \Delta(\mathrm{res}(\delta)) \text{ is decomposable}$$

where res *is the restriction to $k((t))$. There are at most 2 nontrivial factors in any decomposition.*

Proof: Suppose $\delta \in \mathrm{Im}(s)$. Then $\mathbf{i}(\delta) = \mathbf{i}(\mathrm{res}(\delta))$ by (6.0.2). If $\Delta(\delta)$ is decomposable, then the basic theory shows that the index of the factors in any nontrivial decomposition $\Delta(\delta) \cong \Delta(\delta_1) \otimes \Delta(\delta_2)$ can only go down on restriction to $k((t))$. That is, $\mathbf{i}(\mathrm{res}(\delta_i)) \mid \mathbf{i}(\delta_i)$. Yet since $\mathrm{res}(\delta) = \mathrm{res}(\delta_1) + \mathrm{res}(\delta_2)$ (since res is a homomorphism), $\mathbf{i}(\mathrm{res}(\delta)) \mid \mathbf{i}(\mathrm{res}(\delta_1)) \cdot \mathbf{i}(\mathrm{res}(\delta_2))$ by (1.3.1). Therefore the indexes of the factors of δ don't go down at all on restriction to $k((t))$, hence $\Delta(\mathrm{res}(\delta))$ is decomposable. Since each $\Delta(\mathrm{res}(\delta_i))$ is indecomposable (by Theorem 2.2), each $\Delta(\delta_i)$ is indecomposable over $k(t)$ by the same argument, so there can be at most 2 factors in any decomposition of $\Delta(\delta)$.

Conversely, suppose $\Delta(\mathrm{res}(\delta))$ is decomposable, where $s(\mathrm{res}(\delta)) = \delta$, and $\Delta(\mathrm{res}(\delta)) \cong \Delta(\mathrm{res}(\delta_1)) \otimes \Delta(\mathrm{res}(\delta_2))$ is a nontrivial decomposition in $\mathrm{Br}(k((t)))$ (where the δ_i are defined by $s(\mathrm{res}(\delta_i)) = \delta_i$). Then $\delta = \delta_1 + \delta_2$, since s is a homomorphism. Since s preserves index, $\mathbf{i}(\delta) = \mathbf{i}(\delta_2)\,\mathbf{i}(\delta_1)$. Therefore $\Delta(\delta)$ is decomposable. ∎

Corollary 6.4: *There exist indecomposable $k(t)$-division algebras with unequal period and index. For any number field k, there are examples with $(\mathbf{o}, \mathbf{i}) = (p^2, p^3)$, for any prime p.*

Proof: This follows directly from Proposition 6.3, applied to Corollary 3.2: Let δ be the element in the image under s of the indecomposable example of period

p^2 and index p^3 in Corollary 3.2. Then by (6.0.2) and (6.0.3), $\mathbf{o}(\delta) = p^2$ and $\mathbf{i}(\delta) = p^3$. By Proposition 6.3, $\Delta(\delta)$ is indecomposable. ∎

Remarks 6.4.1: (i) This example solves open problem #7 posed by Saltman in [Sa2].

(ii) The author does not know whether these examples are the smallest that exist over $k(t)$, i.e., whether there exist indecomposable $k(t)$-division algebras of period p and index greater than p, when k is a number field.

COROLLARY 6.5: *Suppose $\delta \in \mathrm{Br}(k(t))$ has p-power index, and is in the image of s in (6.0.1). Then*

$$\Delta(\delta) \text{ is decomposable} \iff p \,\big|\, \underline{\mathrm{c}}(\delta).$$

Proof: By Definition 6.2 and Lemma 6.1, $\underline{\mathrm{c}}(\delta)$ for $\delta \in \mathrm{Br}(k(t))$ is the same as $\underline{\mathrm{c}}(\mathrm{res}(\delta))$, where res is restriction to $k((t))$. Therefore this result follows directly from Proposition 6.3. ∎

Using Definition 6.2, Lemma 6.1, and Proposition 6.3, one can attempt to reconstruct all of the proofs of §§2, 3 and 5 in the current setting, as has been done in the above corollaries. The author claims that there is no obstruction to doing this. Proposition 6.3 shows that any decomposition that can be constructed over $k((t))$ can be defined over $k(t)$, and conversely a $k(t)$-division algebra is decomposable if and only if it has a decomposition into factors from the image of s.

As for Section 4, it is not hard to show that results analogous to Theorem 4.1 can be proved using Proposition 6.3. However, the counterexample in Theorem 4.3 and Corollary 4.4 may fail since the indecomposable division algebra presented there might conceivably embed properly in a division algebra representing some element of the kernel C of (6.0.1).

Some questions raised by these results:

(1) Does a decomposition of more than 2 nontrivial factors of prime power index exist over $k(t)$?
(2) Do there exist indecomposable $k(t)$-division algebras of prime period not equal to index?
(3) Does every indecomposable $k(t)$-division algebra of p-power index embed in a larger one of p-power index when k is a number field?

(4) If $\delta \in \mathrm{Im}(s)$, do there exist nontrivial decompositions of $\Delta(\delta)$ with terms in C?

(5) What is the situation over the field $k((s))((t))$ of iterated formal power series in two variables? Is there a nice "geometric" criterion for decomposability there?

References

[Al] A. A. Albert, *Normal division algebras of degree 4 over an algebraic field*, Transactions of the American Mathematical Society **34** (1932), 363–372.

[ART] S. A. Amitsur, L. H. Rowen and J.-P. Tignol, *Division algebras of degree 4 and 8 with involution*, Israel Journal of Mathematics **33** (1979), 133–148.

[AT] M. Artin and J. Tate, *Class Field Theory*, Addison-Wesley, Reading, Mass., 1967.

[Br1] E. Brussel, *Noncrossed products and nonabelian crossed products over $Q(t)$ and $Q((t))$*, American Journal of Mathematics **117** (1995), 377–394.

[Br2] E. Brussel, *Division algebra subfields introduced by an indeterminate*, Journal of Algebra (to appear).

[FSS] B. Fein, D. Saltman and M. Schacher, *Crossed products over rational function fields*, Journal of Algebra **156** (1993), 454–493.

[FS1] B. Fein and M. Schacher, *Crossed products over algebraic function fields*, Journal of Algebra (preprint).

[FS2] B. Fein and M. Schacher, *$\mathbb{Q}(t)$ and $\mathbb{Q}((t))$-admissibility of groups of odd order*, Proceedings of the American Mathematical Society (preprint).

[Jb] B. Jacob, *Indecomposable division algebras of prime exponent*, Journal für die reine und angewandte Mathematik **413** (1991), 181–197.

[JW] B. Jacob and A. R. Wadsworth, *Division algebras over henselian fields*, Journal of Algebra **128** (1990), 126–179.

[Jn] N. Jacobson, *P.I. Algebras, an Introduction*, Lecture Notes in Mathematics **441**, Springer-Verlag, New York, 1975.

[Na] T. Nakayama, *Divisionsalgebren über diskret bewerteten perfekten Körpen*, Journal für die reine und angewandte Mathematik **178** (1938), 11–13.

[Ne] J. Neukirch, *Class Field Theory*, Springer-Verlag, New York, 1982.

[Re] I. Reiner, *Maximal Orders*, Academic Press, London, 1975.

[Ro] L. H. Rowen, *Cyclic division algebras*, Israel Journal of Mathematics **41** (1982), 213–234.

[Sa1] D. Saltman, *Indecomposable division algebras*, Communications in Algebra **7** (1979), 791–817.

[Sa2] D. Saltman, *Finite dimensional division algebras*, in *Azumaya Algebras, Actions, and Modules* (D. Haile and J. Osterburg, eds.), Contemporary Mathematics Vol. 124, American Mathematical Society, Providence, R.I., 1992, pp. 203–214.

[S] O. F. G. Schilling, *The Theory of Valuations*, Mathematical Surveys, No. 4, American Mathematical Society, Providence, R.I., 1950.

[SvdB] A. Schofield and M. van den Bergh, *The index of a Brauer class on a Brauer–Severi variety*, Transactions of the American Mathematical Society **333** (1992), 729–739.

[Se] J.-P Serre, *Local Fields*, Springer-Verlag, New York, 1979.

[Sn1] B. A. Sethuraman, *Indecomposable division algebras*, Proceedings of the American Mathematical Society **114** (1992), 661–665.

[Sn2] B. A. Sethuraman, Private communication.

[Ti] J.-P. Tignol, *Algèbres indécomposables d'exposant premier*, Advances in Mathematics **65** (1987), 205–228.